The Maths E-Book
of
Notes and Examples

In association with

mr barton maths.com

Contents

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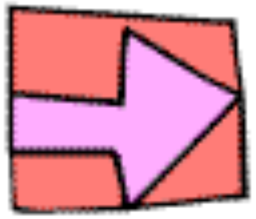
Number



1. Types of Number

1. Integers

"Integer" is just a posh word for whole number.
The thing to remember is that integers can be positive or negative
So: 1, 7, 298, -3, 0 and -49 are all integers, but 2.5 is not!

2. Rational Numbers

Rational Numbers are numbers which can be written as fractions.
Don't Forget: the top and bottom of the fraction (numerator and denominator) must be whole numbers (integers).

So: 4 is a rational number as it can be written as: $4/1$ or $8/2$

0.6 is a rational number as it can be written as: $6/10$ or $3/5$

even 4.285714285714... is a rational number as it can be written as: $30/7$

3. Irrational Numbers

Irrational Numbers are just the opposite of Rational Numbers
They cannot be written as a fraction.
In fact, when these numbers are written in decimal form, the numbers go on forever and ever and the pattern of digits is not repeated.

e.g. The most famous Irrational Number is pi (π), which is 3.1415927..., but √2 and √7 are irrational too

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4. Square Numbers

You can get a Square Number by multiplying any whole number (integer) by itself

So: The first square number is 1, because 1 × 1 = 1.

The second square number is 4, because 2 × 2 = 4, and so on...

The first ten square numbers are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

Look:

You can also get all the square numbers by counting the dots in square patters:

5. Triangle Numbers

You can get all the Triangle Numbers by starting with 1, and then adding 2, then adding 3, then adding 4, and so on...

So: The first triangle number is 1

The second triangle number is 3 (1 + 2)

The third triangle numbers is 6 (1 + 2 + 3)

The first ten triangle numbers are: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55

Look:

You can also get all the triangle numbers by counting the dots in triangle patters:

Challenge:

By looking at the dot patterns, can you see why every time you add together two consecutive triangle numbers, you get a square number?

e.g. 15 + 21 = 36, which is a square number, and 36 + 45 = 81, which is also a square number!

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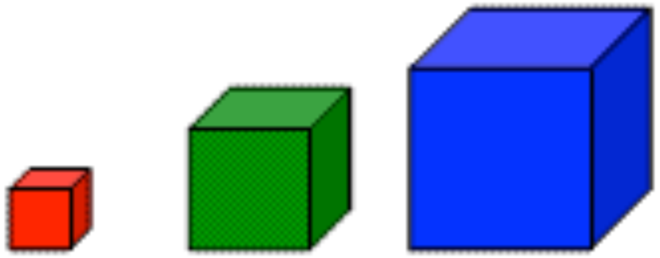
6. Cube Numbers

You can get a Cube Number by multiplying any whole number (integer) by itself and then by itself again.

So: The first cube number is 1, because 1 × 1 × 1 = 1.
The second cube number is 8, because 2 × 2 × 2 = 8, and so on...
The first ten square numbers are: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000

Look:
If Mr Barton was in any way artistic, he would show you that you can get all the cube numbers by counting dots in cubes. Don't take my word for it, try it yourself!



7. Factors

The Factors of a number are all the whole numbers (integers) that divide into your number exactly (there must not be a remainder!)

Don't forget: 1 is a factor of all numbers, and so is the number itself!

e.g. The factors of 12 are: 1, 2, 3, 4, 6 and 12
The factors of 55 are: 1, 5, 11, and 55

Challenge:
Have you any idea why all square numbers seem to have an odd number of factors?

8. Multiples

The Multiples of a number are all the numbers in your number's times table

Don't forget: you must count the number itself!

e.g. Some multiples of 7 are: 7, 14, 21, 28... but there are loads more, like 700 and 4445
Some multiples of 21 are: 21, 42, 63... but there are loads more, like 231 and 1050

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9. Prime Numbers

For some reason, people always get confused with prime numbers, so try to remember this definition and you won't go wrong:

A prime number is a number that has exactly 2 factors, no more, no less

So: 1 is NOT a prime number, as it only has one factor (1)
2 is a prime number as it has two factors (1 and 2)

Don't Forget: 2 is the only EVEN prime number!

7 is a prime number as it has two factors (1 and 7)
21 is NOT a prime number as it has four factors (1, 3, 7 and 21)
1061 is a prime number as it has just two factors (1 and 1061)

Look:

Unfortunately, there does not seem to be any patterns to help us find all the prime numbers, but it's not all bad news. I think the prize for finding the largest prime number is about $1million, and its even more if you find the pattern! If you have a spare 5 minutes, why not give it a go.

Anyway, until then, here are all the prime numbers between 1 and 100:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97.

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2. Prime Factors, HCF & LCM

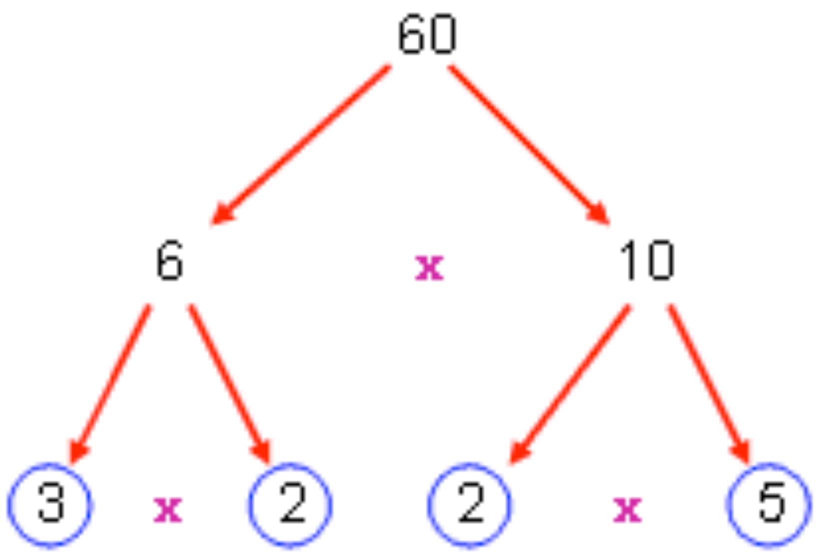
1. Prime Factors

Any positive integer can be written as a product of its prime factors.

Now, that may sound complicated, but all it means is that you can break up any number into a multiplication of prime numbers, and it's really easy to do with Factor Trees!

Don't Forget: **1** is NOT a prime number, so will NEVER be in your factor tree

e.g. Express 60 as a product of its prime factors



3 × 2 × 2 × 5 = 60

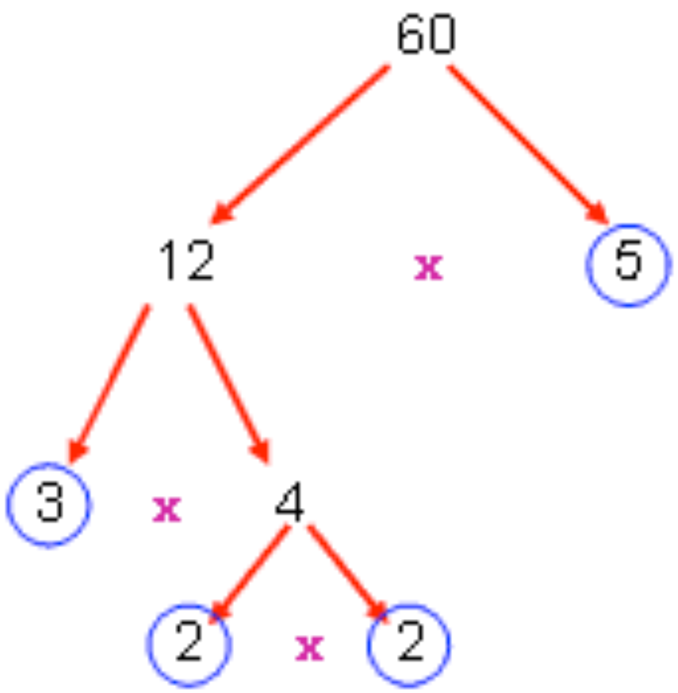
You can break the number up however you like:

6 × 10 or 12 × 5

Continue breaking up each new number into a multiplication

Stop when you reach a Prime Number and put a ring around it

Check your answer by multiplying all the numbers together



3 × 2 × 2 × 5 = 60

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Look: Even though we started a different way, we still ended up with the same answer!
Now, it looks good if you write your answer starting with the smallest numbers:

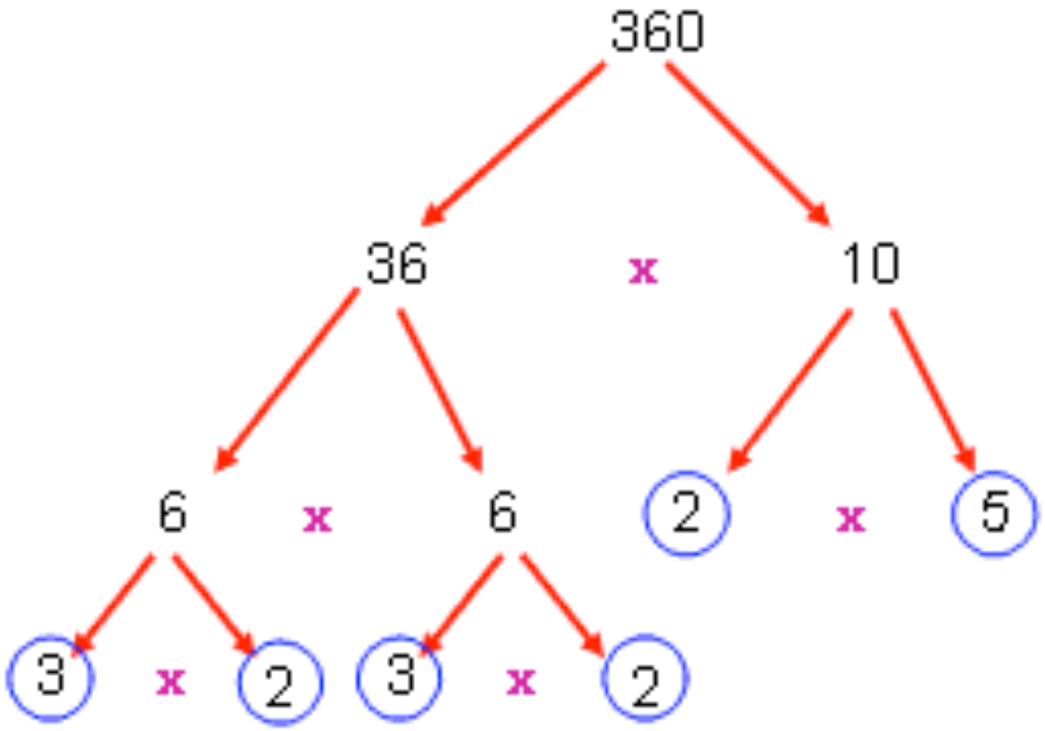
So: $60 = 2 \times 2 \times 3 \times 5$

And if you want to be really posh, you can use indices:

So: $60 = 2^2 \times 3 \times 5$

Now we'll do a harder one, but the technique is just the same.

e.g. Express 360 as a product of its prime factors



You can break the number up however you like. I just went for 36 × 10 because it was easy to spot

Continue breaking up each new number into a multiplication

Stop when you reach a Prime Number and put a ring around it

Check your answer by multiplying all the numbers together

Write the numbers in order

If you can, use indices

$3 \times 2 \times 3 \times 2 \times 2 \times 5 = 360$

$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$

$360 = 2^3 \times 3^2 \times 5$

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2. Highest Common Factor

The Highest Common Factor (HCF) of two numbers, is the highest number that divides exactly into both

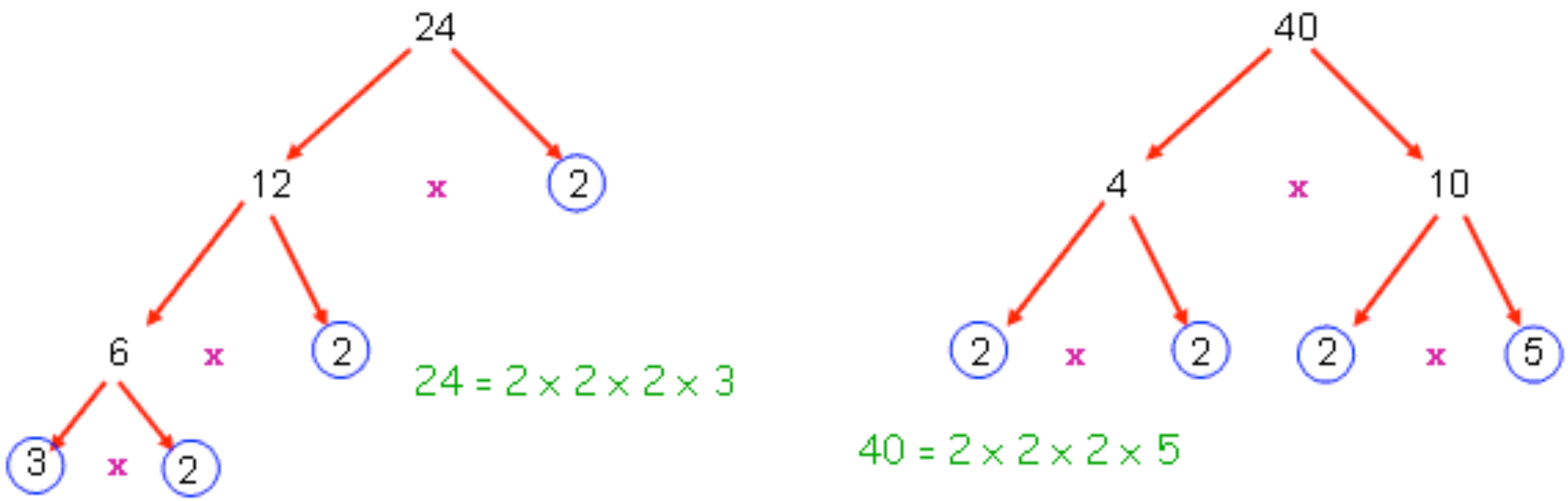
3. Lowest Common Multiple

The Lowest Common Multiple (LCM) of two numbers, is the lowest number that is in the times table of both your numbers

Now, you can find both of these by trial and error, but I will show you a better way!

e.g. Find the LCM and HCF of 24 and 40

First, use Factor Trees to express your numbers as products of their prime factors:



24 = 2 × 2 × 2 × 3

40 = 2 × 2 × 2 × 5

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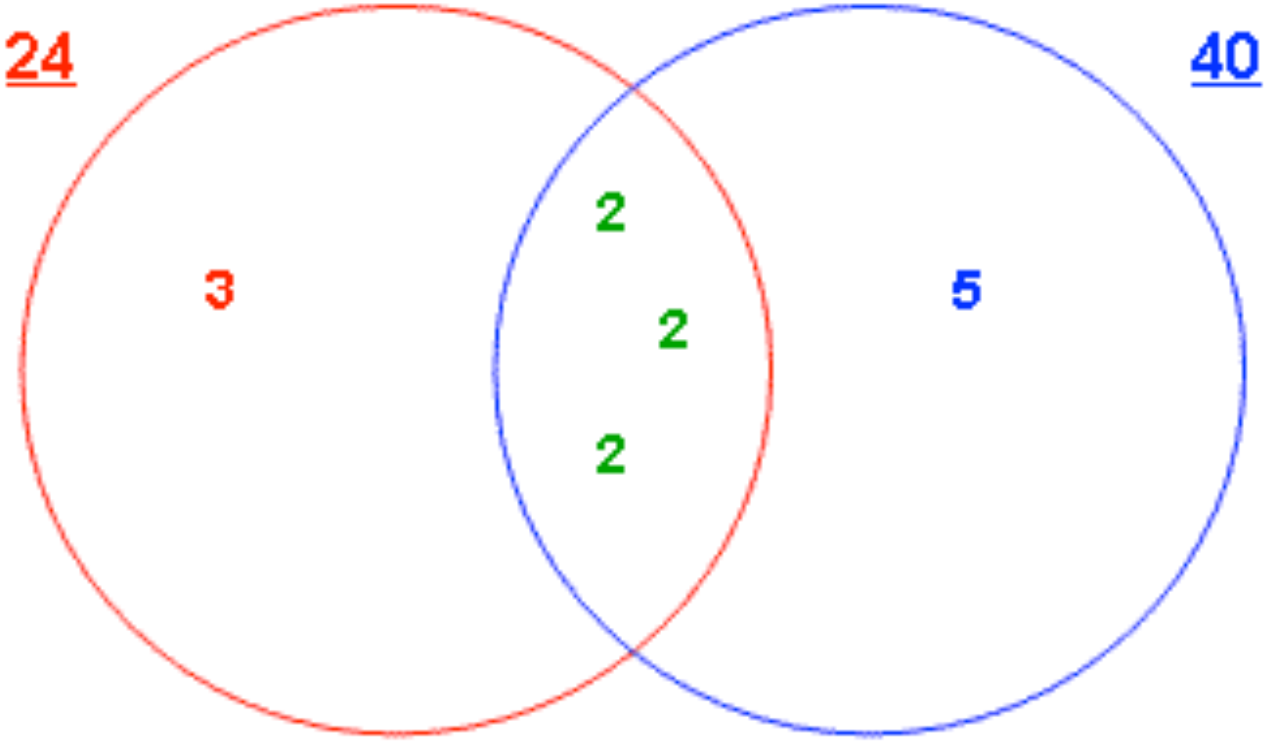
Now, write your answers on top of each other, like this:

24 = 2 × 2 × 2 × 3

40 = 2 × 2 × 2 × 5

Draw two inter-locking circles, and label one 24 and the other 40



Any numbers that appear in both answers go in the middle (the three 2s).

The numbers left over go in the circle they belong to

Now, here is the clever bit:

To get the Highest Common Factor you just multiply all the numbers in the middle

So, HCF = 2 × 2 × 2 = **8**

To get the Lowest Common Multiple you just multiply every number you can see

So, LCM = 3 × 2 × 2 × 2 × 5 = **120**

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3. Bodmas

A question...

What is: **3 + 2 x 4** ?

Now, if you said **20**, then I am afraid you are wrong.
If you try the sum on your calculator - and so long as it is not one of those you get free in a cereal packet - then the answer that should appear on the screen is **11**

But why?...

Well, it's all to do with BODMAS, or BIDMAS depending on which one you prefer.
This is a set of rule which tells you which order you must do operations (like add, divide etc) in order to get questions like the one above correct.

So, what does it stand for?...

B	Brackets	If there are any brackets in your sum, work out what is inside them first. And remember: you must use the rules of Bodmas inside your brackets!

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O or I	Order or Indices	Next up you must look for powers, such as 2^3 and work them out	
D	Divide	Now it's time to sort out your divisions. And remember: divisions can look like this: ÷ or this: $\frac{9}{4}$	
M	Multiply	Next comes the multiplications	×
A	Add	Then add the additions	+
S	Subtract	And last but not least, the subtractions	−

And so long as you follow these rules carefully, you shouldn't go wrong!

But let's go through three examples together...

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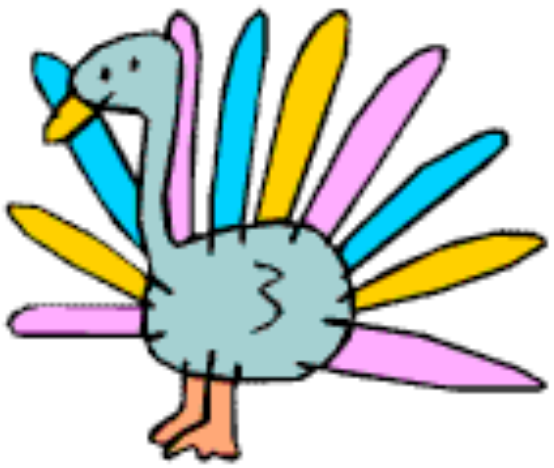
Example 1 – Quite Nice

$$20-(3+2)\times 3$$



1. The first thing we need to do is to sort those brackets out. 3 + 2 = 5, so we are left with this new sum:

$$20-5\times 3$$

2. We have no powers and no divisions, so next up is our multiplication. $5\times 3 = 15$, leaving us with this:

$$20-15$$

3. And now life is easy!

⑤

So, as I hope you can see, all we need to do is break down long, complicated sums into smaller, more manageable ones. And so long as we take our time, and write down each step, we should be okay.

But they do get harder...

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Example 2 – A Bit Trickier

$$3+(2\times 3^2-3)\div 5$$

1. Again, the first thing we need to do is to sort those brackets out. Let's concentrate on them and worry about the rest of the sum later: $\longrightarrow$ $(2\times 3^2-3)$

2. We must make sure we use the same rules of Bodmas inside the brackets. So first we must deal with our power. Remember: 3^2 is 9, not 6! $\longrightarrow$ $(2\times 9-3)$

3. No divisions, so next up is the multiplication: $\longrightarrow$ $(18-3)$

4. Which leaves as a nice subtraction, and tells us the value of our brackets: $\longrightarrow$ 15

5. Now we can return to our original problem, and thankfully it looks a lot nicer: $\longrightarrow$ $3+15\div 5$

6. Keep your brain switched on at this point and remember to do the division first. $\longrightarrow$ $3+3$

7. And even though you might have to go onto two hands to count your fingers, you should get the answer to this one correct $\longrightarrow$ $\textcircled{6}$

Right, are you ready for this one...

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Example 3 – A Nightmare

$$\frac{10+2\times 3}{10-2^3}$$

1. Now, you might not think there are any brackets on this sum… but there are! Whenever the division line goes right across, it is like there are brackets on the top and the bottom, because the whole of the top must be divided by the whole of the bottom:

$$\frac{(10+2\times 3)}{(10-2^3)}$$

2. Right, let sort the top bracket out first:

$$(10+2\times 3)$$

3. Usual deal, multiplication first:

$$(10+6)$$

4. Which means the top of the division is easy enough to work out:

$$16$$

5. Now we have the bottom to deal with:

$$(10-2^3)$$

6. We have to do the power first, and remember, 2^3 is 8, it is definitely not 6!

$$(10-8)$$

7. Which tells us that the bottom of the division is:

$$2$$

8. Which leaves us with a very nice division to do:

$$\frac{16}{2}$$

9. Which finally gives us our answer:

$$\boxed{8}$$

Phew! And if you followed that, you deserve a break!

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4. Rounding and Approximations

A Question...

Imagine you are walking along the street and someone stops you and asks you to do this nice little sum in your head in 30 seconds...

$$\frac{6.0602^2}{3.1092 \times 5.95}$$

If you are anything like my pupils, I can imagine what you might say, and I can't write it on this website...

But, with a little knowledge about rounding and approximations, you should be able to tell that person that the answer is about 2, and then ask them to kindly leave you alone

1. Rounding

Now, there are lots of degrees of accuracy you will need to know how to round to, but the way to tackle any question you could ever possibly be asked is always the same:

1. Circle the last digit you need - what I will call the Key Digit

2. Look at the unwanted digit to the right to it - if it is 5 or above add one on to your Key Digit, if it is less than five, leave your Key Digit alone.

3. Be very careful of the dreaded number 9...

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(a) decimal places

The most common degree of accuracy you are asked to round to is a number of decimal places.

Because mathematicians are lazy, this is normally shortened down to **dp**.

e.g. 5.96 (2dp) means that the answer was probably really long, but when rounded to two decimal places, it was 5.96

The thing you need to remember, and the thing that sounds really obvious, is that if the question asks for **two** decimal places, you must give **two**, no more, no less!

Example 1

Round 5.639 to 1dp

5 . 6 3 9

1. We start by putting a ring around our Key Digit. Now the question has asked for 1 decimal place, so our key digit is the 6, as it occupies the 1st decimal place

2. Next we look at the digit to the right to it – the unwanted number 3. It is less than 5, so we leave the key digit alone.

3. So, to one decimal place, our answer is:

5.6

Example 2

Round 12.0482 to 2dp

1 2 . 0 4 8 2

1. This time the Key Digit is in the 2nd decimal place, which makes it the 4

2. The unwanted digit to the right of it is an 8, which is definitely 5 or above, so we must add one onto our Key Digit

3. So, to two decimal places, our answer is:

12.05

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Example 3

Round 25.72037 to 3dp

2 5 . 7 2 0 3 7

1. This time the Key Digit is in the 3rd decimal place, which makes it the 0

2. The unwanted digit to the right of it is 3, which is definitely less than 5, so just leave our Key Digit alone

3. So, to three decimal places, our answer is:

25.720

Be careful: Some silly people will put 25.72 down as the answer thinking that the 0 makes no difference. But it does! The question has asked for 3dp, so give them 3dp!

Example 4

Round 3.7952 to 2dp

3 . 7 9 5 2

1. This time the Key Digit is in the 2nd decimal place, which makes it the 9

2. The unwanted digit to the right of it is a 5, which is 5 or above, so we must add one onto our Key Digit

But: if we add one to our key digit, we get 10! So, we must add one to the next digit as well, which is the 7

3. So, to two decimal places, our answer is:

3.80

(b) nearest whole, 10, 100, 1000 etc

These are the nicest types of rounding questions, and so long as you have your brain switched on, you shouldn't get too many of them wrong. But don't get cocky, as you can easily make mistakes!

Remember: the size of your rounded number should be a similar size to the number in the question, and you must use zeros to help you with this

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Example 1

Round 3.825 to the nearest whole number

③ . 8 2 5

1. Our Key Digit is always the degree of accuracy the question asks for, which in this case is whole numbers, so we need the 3.

2. The unwanted digit to the right of it is 8, which is definitely more than 5, so we add one to our Key Digit.

3. So, to the nearest whole number, our answer is:

4

Example 3

Round 4,365,901 to the nearest thousand

4 3 6 ⑤ 9 0 1

1. We want the nearest thousand, so our Key Digit must be the number that represents the thousands which is the 5

2. The unwanted digit to the right of it is 9, which is definitely more than 5, so we add one to our Key Digit.

3. So, to the nearest whole number, our answer is:

4,365,000

Example 2

Round 32,825.2 to the nearest hundred

3 2 ⑧ 2 5 . 2

1. We want the nearest hundred, so stick the ring around the digit in the hundreds column, which is the 8.

2. The unwanted digit to the right of it is a 2, which is less than 5, so we leave our Key Digit alone.

3. So, to the nearest hundred, our answer is:

32,800

Example 4

Round 3,999 to the nearest ten

3 9 ⑨ 9

1. We want the nearest ten, so the Key Digit must be the 9 in the tens column

2. The unwanted digit to the right of it is a 9, so we add one on, but we then need to add one on the next 9, and then the 3!

3. So, to the nearest ten, our answer is:

4,000

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(c) significant figures

In higher level SATs, and especially in GCSE and A Level, the nasty examiners are obsessed with Significant Figures.

Again, there is a lazy way of writing this, which is **sf** or **sig fig**.

Crucial: The first significant figure is always the first non-zero number you come across. The second significant figure is the number to the right of that, and so on…

Remember: the size of your rounded number should be a similar size to the number in the question, and you must use zeros to help you with this.

Example 1

Round 28.53 to 1 sig fig

②8 . 5 3

1. The Key Digit has the be the first significant figure, which must be the 2, as it is the first non-zero number

2. Now we carry on as normal looking to the number to the right, which is an 8, so we add one on.

3. So, keeping the size of the answer the same as the question with a zero, to 1 sig fig the answer must be:

30

Example 2

Round 5,322 to 2 sig figs

5③2 2

1. The Key Digit is in the place of the 2nd significant figure, which is the 3

2. The unwanted digit to the right of it is 2, which is definitely less than 5, so we leave our Key Digit alone

3. So, again using zeros to help us, to two sig figs, our answer is:

5300

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Example 3

Round 0.027 to 1 sig fig

0 . 0 2 7

1. Our first significant figure is the first non-zero number, which means it's the 2

2. The unwanted digit to the right of it is 7, so we add one to our Key Digit.

3. No need for extra zeros here, so to the 1 significant figure our answer is:

0.03

Example 5

Round 4.0004 to 2 sig figs

4 . 0 0 0 4

1. The 1st sig fig is the 4, and so the 2nd is the 0 (it is after the 4, so it's significant).

2. The unwanted digit to the right of it is 0, which is definitely less than 5, so we leave our Key Digit alone

3. So, to 2 sig figs, our answer is:

4.0

Example 4

Round 305,216 to 3 sig figs

3 0 5 2 1 6

1. The 1st sig fig is the 3, the 2nd is the 0 (it is after the 3, so it's significant), so the Key Digit is the 5

2. The unwanted digit to the right of it is a 2, so we leave our Key Digit alone.

3. We need some zeros to make our answer the correct size, so to 3 sig figs::

305,200

Example 6

Round 0.089722 to 2 sig figs

0 . 0 8 9 7 2 2

1. Our 1st non zero number is the 8, so the Key Digit must be the 9.

2. The unwanted digit to the right of it is a 7, so we add one on, but that gives us 10, so we must add one to our 8 as well.

3. Keeping our answer the right size, we have:

0.090

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2. Approximations

Right, now that we are experts at rounding, we can use our skills to find approximate answers (or estimates) to horrible looking questions like the one at the start:

$$\frac{6.0602^2}{3.1092\times5.95}$$

Now, you would need to be a bit of a freak to do this in your head, but if you were to round each number to the nearest whole number, or 1 significant figure then you get:

$$\longrightarrow \frac{6^2}{3\times6}$$

And if you now use our BODMAS skills, you should be able to say:

$$\frac{6^2}{3\times6} \longrightarrow \frac{36}{3\times6} \longrightarrow \frac{36}{18} \longrightarrow \boxed{2}$$

The actual answer on the calculator is pretty close: 1.98521838...

So, if we want to sound clever, we can say that:

$$\frac{6.0602^2}{3.1092\times5.95} \approx 2$$

Where the funny sign means: "approximately equal to"

So, always look out for ways to use your rounding skills to turn tricky looking sums into pretty easy ones!

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5. Decimals

Things you might need to be able to do with decimals…

This will vary with your age and what maths set you are in, but here is a list of some of the things you might need to be able to do with decimals:

1. Know the three types of decimals
2. Know how to add, subtract, multiply and divide using decimals
3. Understand the relationship between fractions, decimals and percentages
4. Know how to convert a recurring decimal into a fraction.

And if you are sitting comfortably, then I will begin…

1. The Three Types of Decimal

Here they are…

(a) Exact or Terminating

These are decimals that stop.

They do not go on forever, and so you can write down all their digits

e.g. 0.5, 0.276, 0.523894, 0.0000000004

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(b) Recurring

These are decimals that go on forever and ever, but some of the digits repeat.

e.g. 0.3333333...   0.45454545454545...,   0.14891489148914891489...,

It is impossible to write down all the digits, so we use the dot notation

We place a dot (or two dots) to show which digits repeat

Examples

0 . 5 5 5 5 5 5 5 5 5 5 5...	Here, only the 5 repeats, so we place the dot like this:	→	$0.\dot{5}$
0 . 1 4 3 1 4 3 1 4 3 1...	Now it's the 1 4 3 repeating, so the following is needed:	→	$0.\dot{1}4\dot{3}$
0 . 3 7 7 7 7 7 7 7 7 7 7...	The 7 repeats, but the 3 does not, so it would be wrong to put a dot on the 3. We need this:	→	$0.3\dot{7}$
0 . 8 2 3 4 1 3 4 1 3 4 1...	Here, only the 3 4 1 is repeating, so place the dots at the beginning and end of that group	→	$0.82\dot{3}4\dot{1}$

(c) Irrational

These are what I call dodgy decimals

They go on forever and ever, but the digits do not repeat in a regular pattern

e.g. pi, which is 3.14159265358979 32...

Watch Out: Your calculator only has so much space, so some recurring or irrational decimals might look like the actually terminate!

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2. Working with Decimals

If you can add, subtract, multiply and divide whole numbers (integers), then you should be okay with decimals, so long as you are careful!

(a) Adding and Subtracting

The key thing here is to write the numbers out on top of each other and line up your decimal points. That way you won't make a daft mistake

Example 1

12.875 + 0.34

Write the numbers out on top of each other, and line up your decimal points

$$\begin{array}{rr} & 12.875 \\ + & 0.34 \\ \hline \end{array}$$

Now, so long as you remember how to add, and be careful when carrying numbers, you should get the answer of:

13.215

Example 2

0.62 - 0.0159

Again, write the numbers out on top of each other, and line up your decimal points

$$\begin{array}{rr} & 0.6200 \\ - & 0.0159 \\ \hline \end{array}$$

Sometimes I find it easier to fill in zeros on the top line to make subtracting easier.

and remember all your rules from primary school about borrowing from the number to the left

You should get: **0.6041**

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(b) Multiplying

There are a few different ways of doing this, so feel free to ditch my method if you find a better one, but basically if there is a whole number involved I do the sum as normal, and if there are two decimals, then I change the question to make it easier!

Example 1

7 × 1.36

Write the numbers out on top of each other, putting the whole number on the bottom

Remember: it doesn't matter which order you do multiplications

$$\begin{array}{r} 1.36 \\ \times \quad 7 \\ \hline \end{array}$$

Now, just multiply each digit in turn by 7, and remember to carry your numbers, and you should end up with:

9.52

Example 2

0.32 × 0.528

Now, I don't like the look of this at all, but if I multiply the 0.32 by 100 and the 0.528 by 1000, then I get a much easier sum:

32 × 528

Now, I do these kind of sums using the grid method, but feel free to use your own way:

x	500	20	8
30	15000	600	240
2	1000	40	16

$$\begin{array}{r} 15000 \\ 1000 \\ 600 \\ 240 \\ 40 \\ + \quad 16 \\ \hline 16896 \end{array}$$

Now, the answer to this question is 16,896, but to get the answer to the original question we must undo our changes, so divide by 100 and then divide by 1000

Which gives us: **0.16896**

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(c) Dividing

Fortunately, questions about dividing with decimals do not come up all that often, but whenever they do I again use the same method:

1. Make the question easier by multiplying the numbers by 10, 100, or 1000

2. Do the easier sum

3. Remember to undo your changes by multiplying or dividing by 10, 100, or 1000 to get the actual answer

Example 75.92 ÷ 1.3

Right, let's sort those horrible numbers out first.

Multiply the 75.92 by 100 and the 1.3 by 10, and things should look a whole lot nicer → **7592 ÷ 13**

Now, it all depends how you like to do these. I write it out like this:

$$13\overline{)7592}$$

And then I do a lot of talking to myself like this:

- How many 13s go into 7?... None, so carry the seven
- How many 13s go into 75?... Erm… 5, remainder 10, so carry the 10
- How many 13s go into 109?... Erm… erm… 8, remainder 5
- How many 13s go into 52?... 4 exactly!

$$\begin{array}{r} 0584 \\ 13\overline{)7592} \end{array}$$

→ So, our answer is 584, but remember we must undo our changes →

Well, multiplying 75.92 by 100 made the answer 100 times too big, so we must divide by 100, BUT: multiplying 1.3 by 10 made the answer 10 times too small (as we were dividing), so we must multiply by 10

Which gives us: → **5 8 . 4**

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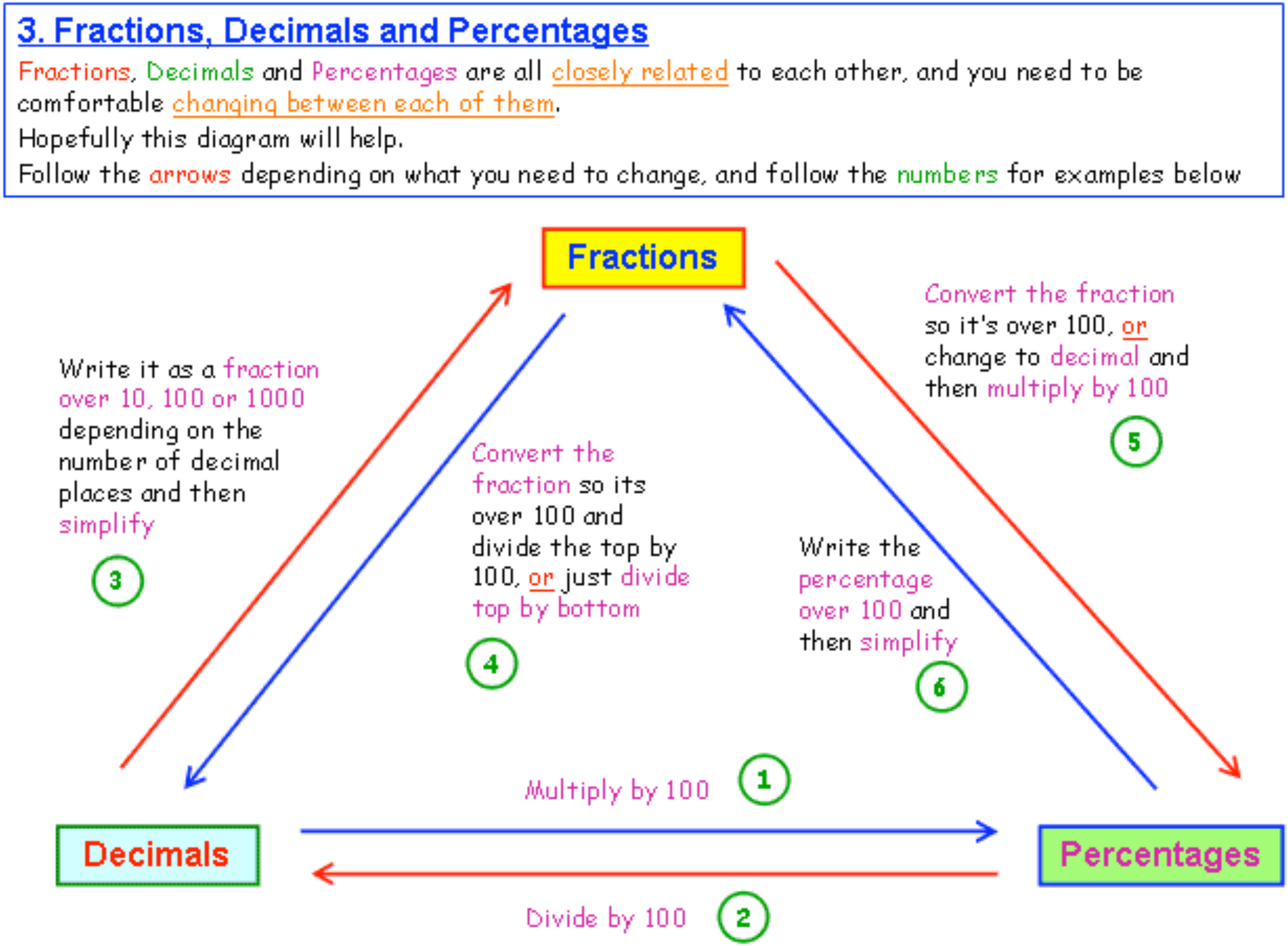

3. Fractions, Decimals and Percentages

Fractions, Decimals and Percentages are all closely related to each other, and you need to be comfortable changing between each of them.

Hopefully this diagram will help.

Follow the arrows depending on what you need to change, and follow the numbers for examples below

Write it as a fraction over 10, 100 or 1000 depending on the number of decimal places and then simplify ③

Convert the fraction so its over 100 and divide the top by 100, or just divide top by bottom ④

Convert the fraction so it's over 100, or change to decimal and then multiply by 100 ⑤

Write the percentage over 100 and then simplify ⑥

Multiply by 100 ①

Divide by 100 ②

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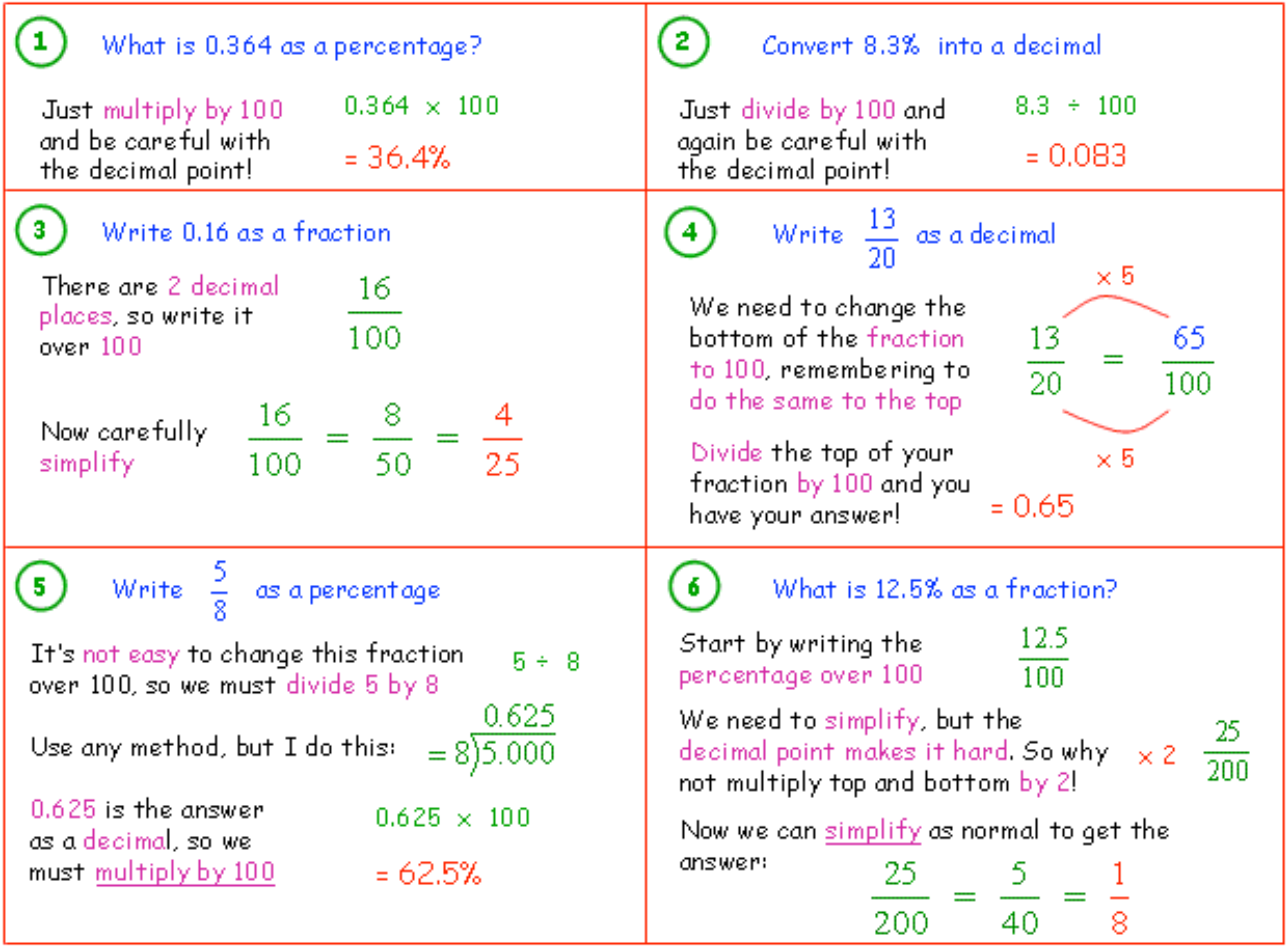

① What is 0.364 as a percentage?

Just multiply by 100 and be careful with the decimal point!

0.364×100

$= 36.4\%$

② Convert 8.3% into a decimal

Just divide by 100 and again be careful with the decimal point!

$8.3 \div 100$

$= 0.083$

③ Write 0.16 as a fraction

There are 2 decimal places, so write it over 100

$\frac{16}{100}$

Now carefully simplify

$\frac{16}{100} = \frac{8}{50} = \frac{4}{25}$

④ Write $\frac{13}{20}$ as a decimal

We need to change the bottom of the fraction to 100, remembering to do the same to the top

$$\frac{13}{20} \overset{\times 5}{=} \frac{65}{100} \quad \times 5$$

Divide the top of your fraction by 100 and you have your answer!

$= 0.65$

⑤ Write $\frac{5}{8}$ as a percentage

It's not easy to change this fraction over 100, so we must divide 5 by 8

$5 \div 8$

Use any method, but I do this:

$= 8\overline{)5.000}$ with 0.625 above

0.625 is the answer as a decimal, so we must multiply by 100

0.625×100

$= 62.5\%$

⑥ What is 12.5% as a fraction?

Start by writing the percentage over 100

$\frac{12.5}{100}$

We need to simplify, but the decimal point makes it hard. So why not multiply top and bottom by 2!

$\times 2 \quad \frac{25}{200}$

Now we can simplify as normal to get the answer:

$\frac{25}{200} = \frac{5}{40} = \frac{1}{8}$

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4. Convert a Recurring Decimal into a Fraction

As I hope you've seen, its not too bad to convert fractions into decimals, or terminating decimals into fractions, but what about recurring decimals that go on forever?

Warning: This is hard! But bare with me and I'll try to take you through an example:

Example Convert 0.165165165165... into a fraction

1. We start with a bit of algebra:

Let x = 0.165165165165....

2. We want to move the decimal point to the right so that the repeated block of digits appears in front of the decimal point, so... to move the point 3 places we must multiply by 1000!

0.165165165165... × 1000 = 165.165165165...

3. Now, if 0.165165165... = x, then 165.165165165... must equal... 1000x!

1000x = 165.165165165165...

x = 0.165165165165...

4. This is the clever/hard bit. If we subtract the top from the bottom we get:

1000x - x = 999x

165.165165165 - 0.165165165 = 165

999x = 165

5. So, by dividing both sides by 999 we get:

$$x = 165/999$$

Which, if you check on a calculator is correct!

Challenge: When you convert some fractions into decimals, such as $7/8$ $3/10$ $9/16$ $11/50$ you get a terminating decimal, but with others such as $2/3$ $6/11$ $5/12$ $3/20$ $8/41$ You get a recurring decimal! What is the rule?

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6. Fractions

Things you might need to be able to do with fractions...

This will vary with your age and what maths set you are in, but here is a list of some of the things you might need to be able to do with fractions:

1. Know the all important fraction's lingo
2. Know how to find the fraction of a quantity
3. Know all about equivalent fractions so you can simplify
4. Understand proper and improper fractions
5. Know how to add, subtract, multiply and divide using decimals
6. Understand the relationship between fractions, decimals and percentages

And so, without further ado, let's get on with it...

1. Fraction's Lingo

Right, let's try and stop talking about the top and the bottom of a fraction, and instead go for something a bit fancier, like this:

numerator

$$\frac{3}{4}$$

denominator

Warning: every now and again, Mr Barton forgets to call them this, so watch out for "top" and "bottom" creeping into these notes

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2. Fraction of a Quantity

There is a simple method which always works with these types of questions:

1. Divide by the bottom (finds you the value of one fraction)

2. Multiply by the top (gives you the value of the number of fractions you need)

Example 1

What is $\frac{3}{4}$ of 24?

1. If you divide the quantity (24) by the denominator of the fraction (4), it tells you the value of $\frac{1}{4}$

$24 \div 4 = 6$ so, $\frac{1}{4} = 6$

2. But we don't want $\frac{1}{4}$, we want $\frac{3}{4}$

so, we must multiply our answer (6) by the numerator (3)

$6 \times 3 = 18$ so, $\frac{3}{4} = \boxed{18}$

Example 2

Find $\frac{5}{7}$ of 2.436 Kg
(give your answer in grams)

Now, if before we start, let's change 2.436 Kg into grams so we get nicer numbers and so we are ready for our answer:

$2.436 \times 1000 = 2436$ so, 2.436kg = 2436g

Now we just do the same as before:

1. Divide by the bottom (2436 ÷ 7)

2. Multiply our answer by the top (× 5)

$\frac{1}{7} = 348$ so, $\frac{5}{7} = 1740$

Remember: give units in your answer:

1740g

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3. Equivalent Fractions

Equivalent fractions are just fractions which have exactly the same value.

You need good knowledge of equivalent fractions when simplifying your answers, and also when adding and subtracting fractions.

Here is the rule:

Whatever you multiply or divide the top by, do the exact same to the bottom!

Example 1

$$\frac{2}{7} = \frac{?}{21}$$

Ask yourself: "what has been done to the 7 to make it 21?"

And then do the same to the top!

$$\begin{tikzcd} \frac{2}{7} \arrow[rr, bend left, "\times 3"] \arrow[rr, bend right, "\times 3"'] & = & \frac{6}{21} \end{tikzcd}$$

Example 2

$$\frac{49}{70} = \frac{7}{?}$$

Ask yourself: "what has been done to the 49 to make it 7?"

And then do the same to the bottom!

$$\begin{tikzcd} \frac{49}{70} \arrow[rr, bend left, "\div 7"] \arrow[rr, bend right, "\div 7"'] & = & \frac{7}{10} \end{tikzcd}$$

Example 3

Simplify: $\frac{48}{54}$

We are looking to make the fraction as simple as possible (i.e. contain the smallest possible whole numbers)

We need a number to divide both the top and the bottom by (a factor of both)

We stop dividing when the top and the bottom do not share any more factors

It doesn't matter how long it takes!

$$\begin{tikzcd} \frac{48}{54} \arrow[rr, bend left, "\div 2"] \arrow[rr, bend right, "\div 2"'] & = & \frac{24}{27} \arrow[rr, bend left, "\div 3"] \arrow[rr, bend right, "\div 3"'] & = & \frac{8}{9} \end{tikzcd}$$

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4. Proper and Improper Fractions

In a proper fraction, the bottom is bigger than the top, like: $\frac{3}{4}$

In an improper fraction (top heavy), the top is bigger than the bottom, like: $\frac{9}{7}$

Sometimes, improper fractions are written as mixed number fractions, like: $3\frac{2}{7}$

You need to be able to switch between improper and mixed number fractions!

Example 1

Write: $^{22}/_{5}$ as a mixed number fraction

Okay, so here we have 22 lots of $^{1}/_{5}$

How many $^{1}/_{5}$ do we need to make one whole?

Well, if you think about a cake sliced into fifths, then we would need 5 slices to make a whole

So, how many wholes can we make out of our 22?

Well, 5 goes into 22… erm… 4 times, with a remainder of… erm… erm… 2!

So, our 22 makes 5 wholes with 2 parts left over

So…

$$\frac{22}{5} = 4\frac{2}{5}$$

Example 2

Write: $3\frac{5}{8}$ as an improper fraction

Right, now we have 3 whole ones, and 5 lots of $^{1}/_{8}$

How many lots of $^{1}/_{8}$ are then in each whole?…

Well, one whole is $^{8}/_{8}$, so there must be 8!

So, how many lots of $^{1}/_{8}$ in our 3 wholes?

$3 \times 8 = 24$!

But remember, we also have our 5 lots of $^{1}/_{8}$

So, altogether we have (24 + 5) lots of $^{1}/_{8}$

So…

$$3\frac{5}{8} = \frac{29}{8}$$

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5. Adding, Subtracting, Multiplying and Dividing Fractions

Warning: this is one of those topics everyone messes up!

Don't mix up your rules for adding and subtracting with those for multiplying and dividing!

(a) Adding and Subtracting

1. Change any mixed number fractions into improper (top heavy) fractions
2. Choose a number that both denominators go into (are factors of)
3. Use your skills of equivalent fractions to make both fractions have that chosen number as their denominator
4. Add/subtract the numerators together, keep the denominator the same, and simplify!

Why can't I just add the tops and the bottom together, cos that'd be dead easy?...

Imagine doing this question $\frac{1}{3} + \frac{1}{5}$

So, you want to add the tops and the bottoms... $\frac{1}{3} + \frac{1}{5} = \frac{2}{8}$

Simplify it: $\frac{2}{8} = \frac{1}{4}$

But look! We started off with $1/3$, we added something to it, we got $1/4$ for our answer, which is smaller than $1/3$

THIS IS ABSOLUTE RUBBISH!!!

(but people still do it!)

Example $3\frac{1}{3} - \frac{4}{5}$

1. Change the mixed number fraction: $3\frac{1}{3} = \frac{10}{3}$

2. Choose a number both denominators are factors of

3 and 5 are both factors of 15

3. Change both fractions so they have 15 on the bottom:

$$\frac{10}{3} \overset{\times 5}{\underset{\times 5}{=}} \frac{50}{15} \qquad \frac{4}{5} \overset{\times 3}{\underset{\times 3}{=}} \frac{12}{15}$$

4. Subtract tops, leave bottoms, simplify:

$$\frac{50}{15} - \frac{12}{15} = \frac{38}{15} = 2\frac{8}{15}$$

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(a) Multiplying and Dividing

Good News: This is a lot easier than adding and subtracting!

How to Multiply fractions:

(1) Change any mixed number into improper (top heavy) fractions

(2) Multiply tops together and multiply bottoms together

(3) Simplify your answer

How to Divide fractions:

(1) Change mixed number fractions

(1) Flip the second fraction upside down

(2) Change the division sign to a multiply

(3) Multiply and simplify!

Example 1

$$\frac{2}{5} \times 1\frac{3}{4}$$

1. Change the mixed number fraction: $1\frac{3}{4} = \frac{7}{3}$

2. Multiply tops together and bottoms together:

$$\frac{2}{5} \times \frac{7}{4} = \frac{2 \times 7}{5 \times 4} = \frac{14}{20}$$

3. Simplify:

$$\frac{14}{20} = \frac{7}{10}$$

Example 2

$$3\frac{2}{7} \div \frac{5}{6}$$

1. Change the mixed number fraction: $3\frac{2}{7} = \frac{23}{7}$

2. Flip the second fraction: $\frac{5}{6} \rightarrow \frac{6}{5}$

3. Change sign to multiply: $\frac{23}{7} \times \frac{6}{5}$

4. Finish it off by multiplying and then simplifying!

$$\frac{23}{7} \times \frac{6}{5} = \frac{23 \times 6}{7 \times 5} = \frac{138}{35} = 3\frac{33}{35}$$

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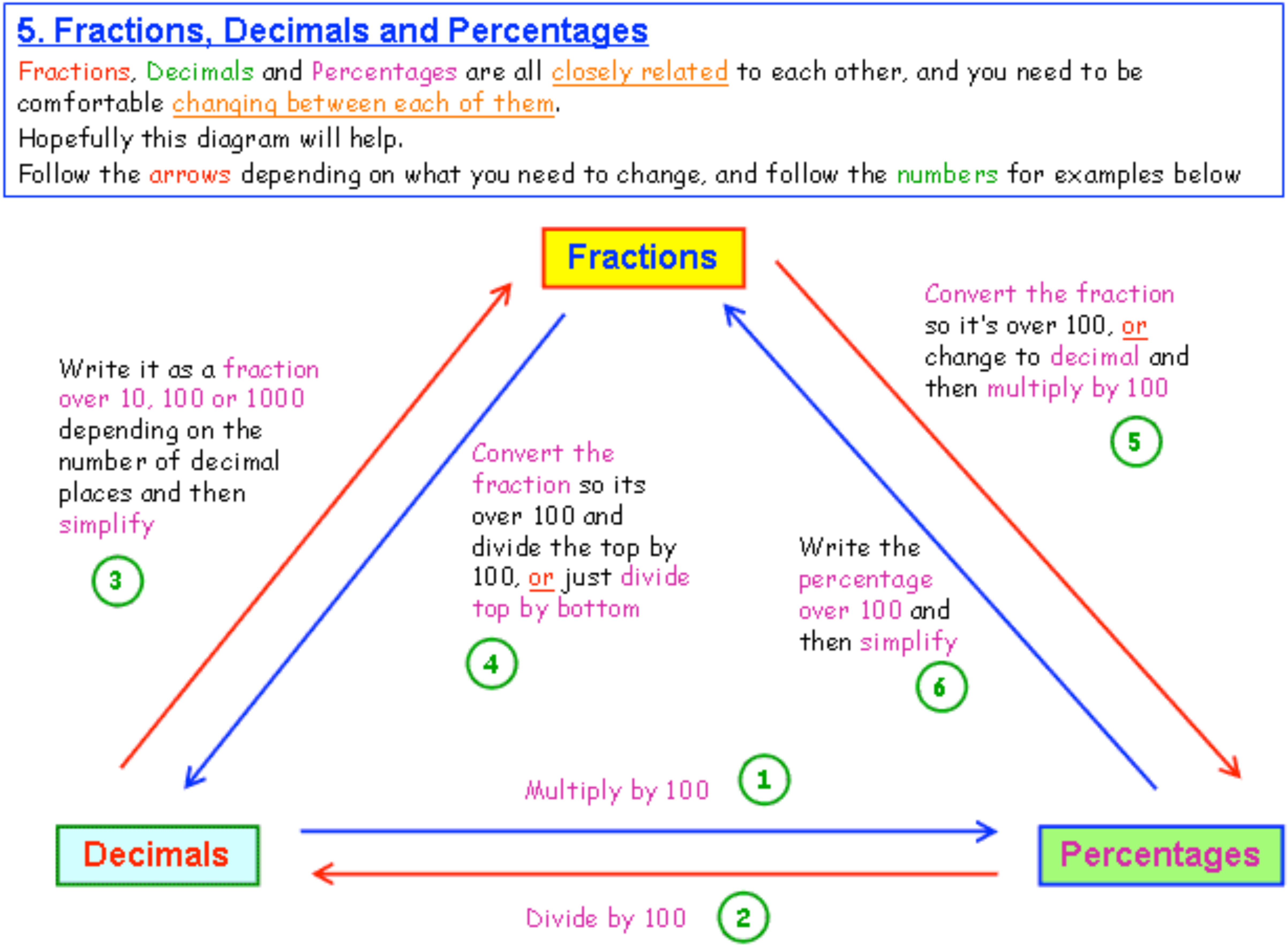

5. Fractions, Decimals and Percentages

Fractions, Decimals and Percentages are all closely related to each other, and you need to be comfortable changing between each of them.

Hopefully this diagram will help.

Follow the arrows depending on what you need to change, and follow the numbers for examples below

Fractions

Write it as a fraction over 10, 100 or 1000 depending on the number of decimal places and then simplify ③

Convert the fraction so its over 100 and divide the top by 100, or just divide top by bottom ④

Convert the fraction so it's over 100, or change to decimal and then multiply by 100 ⑤

Write the percentage over 100 and then simplify ⑥

Multiply by 100 ①

Decimals

Percentages

Divide by 100 ②

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① What is 0.364 as a percentage?

Just multiply by 100 and be careful with the decimal point!

0.364×100

$= 36.4\%$

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Now carefully simplify

$$\frac{16}{100} = \frac{8}{50} = \frac{4}{25}$$

④ Write $\frac{13}{20}$ as a decimal

We need to change the bottom of the fraction to 100, remembering to do the same to the top

$$\overset{\times 5}{\overbrace{\frac{13}{20} = \frac{65}{100}}}$$

$$\underset{\times 5}{\underbrace{\phantom{\frac{13}{20} = \frac{65}{100}}}}$$

Divide the top of your fraction by 100 and you have your answer!

$= 0.65$

⑤ Write $\frac{5}{8}$ as a percentage

It's not easy to change this fraction over 100, so we must divide 5 by 8

$5 \div 8$

Use any method, but I do this:

$$\begin{array}{r} 0.625 \\ = 8\overline{)5.000} \end{array}$$

0.625 is the answer as a decimal, so we must multiply by 100

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We need to simplify, but the decimal point makes it hard. So why not multiply top and bottom by 2!

$$\times 2 \quad \frac{25}{200}$$

Now we can simplify as normal to get the answer:

$$\frac{25}{200} = \frac{5}{40} = \frac{1}{8}$$

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7. Percentages

Things you might need to be able to do with percentages...

This will vary with your age and what maths set you are in, but here is a list of some of the things you might need to be able to do with percentages:

1. Understand what percentages are
2. Find the percentage of an amount in your head and with a calculator
3. Calculate percentage change
4. Calculate percentage increase
5. Calculate percentage decrease
6. Understand compound interest
7. Be able to calculate reverse percentages

It's a big one this one, so without further ado, let's get going...

1. What are percentages?

A percentage is just a fraction whose denominator (bottom) is 100.

So, if we say "32%", what we mean is

$$\frac{32}{100}$$

Now, what is really important before we start is that you understand how to change back and forward from percentages to fractions and decimals.

e.g. You need to able to say that if someone got 23 out of 45 in a test, this is the same as getting 51.1% (1dp)

It's probably best to go back to 6. Fractions, or 5. Decimals, until you are happy with this

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2. Percentage of an Amount

For easy numbers, these sort of questions are fine to do armed merely with a pencil, some paper, and your brain. However, you also need to be able to them on your calculator.

(a) In Your Head

With a bit of knowledge about dividing things by 10 and 100, and some common sense, it's not too bad working out percentage questions in your head.

Example You have £320. What is (a) 15%, (b) 63%, (c) 17.5%

I always start these by writing down the percentages that I know and which might help me:

To find 10% →	Divide by 10 →	320 ÷ 10 = 32 →	10% = £32
To find 1% →	Divide by 100 →	320 ÷ 100 = 3.2 →	1% = £3.20
To find 50% →	Divide by 2 →	320 ÷ 2 = 160 →	50% = £160
To find 20% →	Double 10% →	32 x 2 = 64 →	20% = £64
To find 5% →	Half 10% →	32 ÷ 2 = 16 →	5% = £16
To find 2.5% →	Half 5% →	16 ÷ 2 = 8 →	2.5% = £8

And now we can build up our answers with a bit of simple addition!

(a) 15%

10%	£32
+ 5%	£16
15%	£48

(b) 63%

50%	£160
10%	£32
1%	£3.20
1%	£3.20
+ 1%	£3.20
63%	£201.60

(c) 17.5%

10%	£32
5%	£16
+ 2.5%	£8
17.5%	£56

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(b) On a Calculator

The previous method is fine for easy numbers, and you certainly need to know how to do it for non-calculator exam papers, but if you understand the following method, then life becomes so much easier

It all comes from the fact that percentages are just decimals in disguise.
As we found out in 5. Decimals, to turn a percentage into a decimal we divide by 100.
So, to find a percentage of something:

1. Turn the percentage into a decimal (divide by 100)
2. Multiply your amount by that decimal (use your calculator!)

Example 1

Find 23% of 135g

1. Turn 23% into a decimal:

23 ÷ 100 = 0.23

2. Multiply the amount (135g) by the decimal:

135 × 0.23 = 31.05g

Remember your units!

Example 2

Find 4% of £22.45

1. Turn 4% into a decimal:

4 ÷ 100 = 0.04

It's not 0.4!

2. Multiply the amount (£22.45) by the decimal:

22.45 × 0.04 = £0.90 (2dp)

Notice I have rounded my answer to 2dp as we are dealing with money

Example 3

Find 31.8% of 1,435,988

1. Turn 31.8% into a decimal:

31.8 ÷ 100 = 0.318

2. Multiply the amount (1,435,988) by the decimal:

1,435,988 × 0.318 = 456,644 (nearest whole number)

Here I've chosen to round to the nearest whole number as the numbers were pretty big

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3. Percentage Change

This is a little calculation that everyone forgets. Don't let it happen to you!

You use this when some amount has gone up or down, and you want to know by what percentage it has gone up or down by.

Formula:

$$\text{Percentage Change} = \frac{\text{New value} - \text{Old value}}{\text{Old Value}} \times 100$$

Example 1

After using mrbartonmaths.com, your mark in your maths test went from 34 to 46. What percentage increase is this?

Okay, so we just use the formula:

New Value = 46, Old Value = 34

So :

$$\text{Percentage Change} = \frac{46 - 34}{34} \times 100$$

$$= \frac{12}{34} \times 100$$

= 35.3% (1dp)

Example 2

What a bargain! Scientific calculators have been reduced in price from £4.99 to £3.50. What percentage decrease is this?

Again, so we just use the formula:

New Value = 3.50, Old Value = 4.99

So :

$$\text{Percentage Change} = \frac{3.50 - 4.99}{4.99} \times 100$$

$$= \frac{-1.49}{4.99} \times 100$$

= - 29.9% (1dp)

The minus sign just means it's a decrease

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4. Percentage Increase

Let me see if I can explain how to do these with a little example...

How would you find 23% of something with a calculator?... multiply by 0.23, right?
So, what would you multiply by to increase something by 23%?...
Well, it must have something to do with 0.23, but you need to add 23% onto the original.
Well, the original is 100%, and you need to add 23% onto it, which means you want... 123%
So, to increase something by 23% you need to... multiply by 1.23!

Method for Percentage Increase:
Multiply your amount by (**1** + whatever the percentage is as a decimal)

Example 1

Increase £235 by 17%

Okay, so what do we multiply 235 by?...

Well, 17% as a decimal is 0.17

So, we multiply by (1 + 0.17), which is 1.17!

235 × 1.17

= £274.95

Example 2

Whilst writing this website, my weight has increased by 3.5%. I used to be 87kg. What weight am I now?

Okay, so what do we multiply 87 by?...

Well, 3.5% as a decimal is... erm... 0.035

Be careful with that one!

So, we multiply by (1 + 0.035), which is 1.035!

87 × 1.035

= 90.045kg

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5. Percentage Decrease

Again, let me see if I can explain how to do these with a little example...

How would you find 18% of something with a calculator?... multiply by 0.18, right?

So, what would you multiply by to decrease something by 18%?...

Well, it must have something to do with 0.18, but you need to subtract 18% from the original.

Well, the original is 100%, and you need to subtract 18% from it, which means you want... 82%

So, to decrease something by 18% you need to... multiply by 0.82!

Another way to think about this is once you've taken 18% away, there is only 82% left, and we all know how to find 82%, don't we?...

Method for Percentage Decrease:

Multiply your amount by (**1 -** whatever the percentage is as a decimal)

Example 1

Decrease 250g by 24%

Okay, so what do we multiply 250 by?...

Well, 24% as a decimal is 0.24

So, we multiply by (1 - 0.24), which is 0.76!

Which kind of makes sense, because if you lose 24%, you are left with 76%!

250 × 0.76

= 190g

Example 2

Since I took a break from teaching, my bank balance has dropped by 64.5%. It used to be £10.20. What is it now?

Okay, so what do we multiply 10.20 by?...

Well, 64.5% as a decimal is... erm... 0.645

So, we multiply by (1 - 0.645), which is... let me get my calculator... 0.355!

10.20 × 0.355

= £3.62 (2dp)

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6. Compound Interest

Everyone seems to hate compound interest, but if you understood Sections 4 and 5, then you are going to be flying! Try to follow this…

Example

The bank pays me a compound interest rate of 5% on my balance each year. At the start I have £30 in there. How much do I have after 25 years?

Now, what we definitely DO NOT do is to work out what 5% of £30 is, and the multiply this by 25!

Why?… Well, because you don't just earn 5% on the £30, you earn it on however much is in your bank at the end of each year, which is always growing!

Right, so each year my balance increases by 5%. Let's see what we've got…

End of Year **1**	→	30×1.05	→	£31.50
End of Year **2**	→	31.50×1.05	→	£33.075
End of Year **3**	→	33.075×1.05	→	£34.72875
End of Year **4**	→	34.72875×1.05	→	£36.4651975

Now, if we kept going like this, we would get the right answer, but it's going to take all day, and look at the size of the numbers! I can't be bothered writing all them down. No way.

But, think about what we are actually doing…

30×1.05… get the answer… $\times 1.05$… get the answer… $\times 1.05$… get the answer… $\times 1.05$… etc…

Which is just: $30 \times 1.05 \times 1.05 \times 1.05 \times 1.05 \times$…

How many times do we need to multiply by 1.05?… 25 times!

So, to work it out a quick way we can just do this sum: 30×1.05^{25}

And if you are good on your calculator, you should get: £101.59 (2dp)

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7. Reverse Percentages

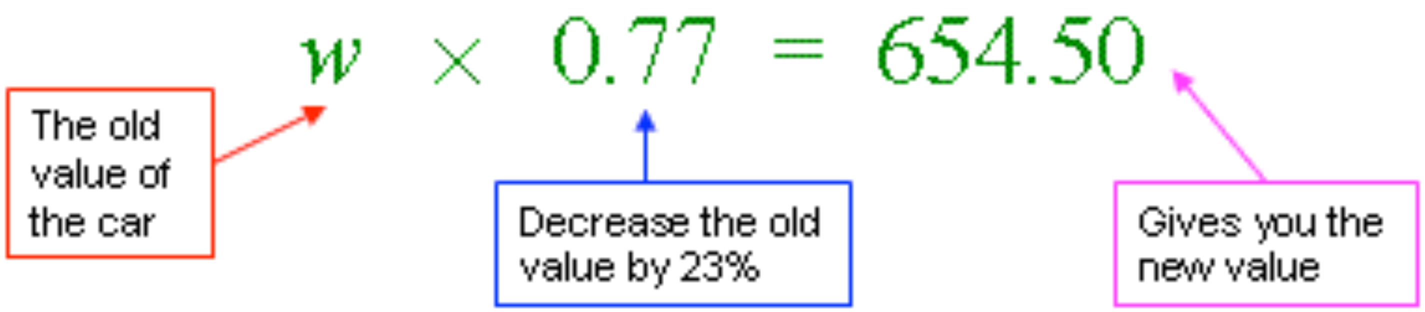
I'll get straight to the point: no-one spots these, everyone messes them up, but if you are careful, and if you try your best to follow this example, you'll be fine!

Example

For some strange reason, ever since I decorated it with "I Love Maths" stickers, my car has gone down in value by 23%. It is now worth £654.50. How much was it worth before?

Now, the key to spotting questions like this are words such as "used to", "old" and "before" - words that suggest you need to work out something that happened in the past.

Think about this: my car used to be worth a certain amount (call it w), then it went down in value by 23%, and it is now worth £654.50. I think I can write that information like this:



$$w \times 0.77 = 654.50$$

The old value of the car

Decrease the old value by 23%

Gives you the new value

So now all we need to do is work out the value of w!

Well, if you divide both sides of the equation by 0.77 you get...

$$w = \frac{654.50}{0.77}$$

And so, using my calculator, w must equal... 850

So my car used to be worth £850

And the beauty of these questions is that you can check your answer by going back to the question:

The car used be worth £850, it's value fell by 23%.

Well... 850 × 0.77 = ... wait for it... £654.50, which is exactly what we were hoping for!

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8. Negative Numbers

WARNING

If you are not concentrating, negative numbers can trip up the best of mathematicians. So... have a glass of water, shake all other thoughts out of your head, sit down, take a deep breath, and let's begin...

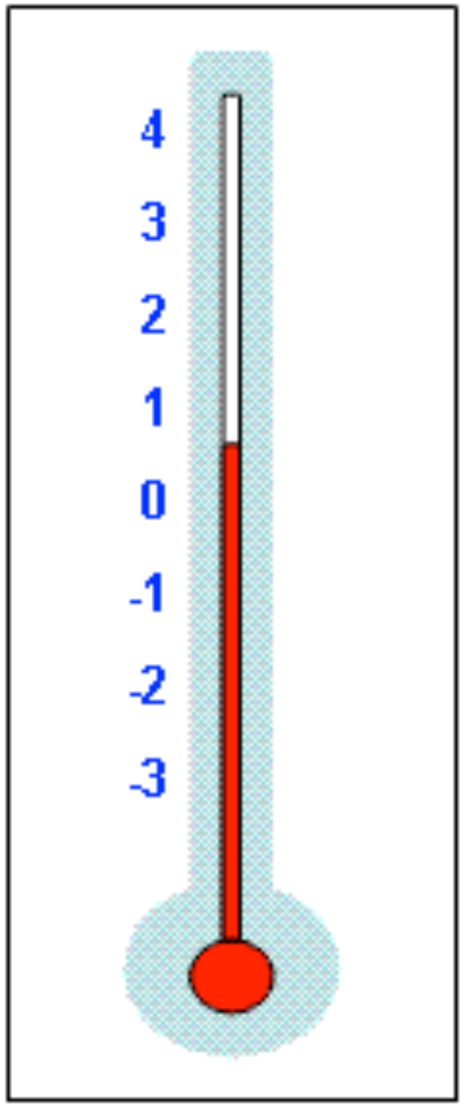
The Number Line

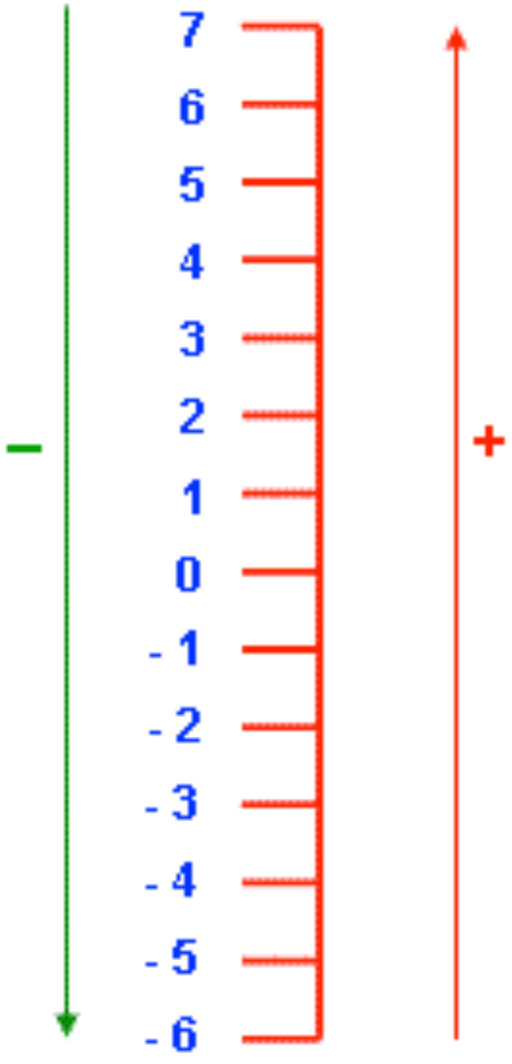
The key to negative numbers is the number line.

Now, I like to think of the number line going up and down, so when you add you go up, and when you subtract, you go down. Kind of like a thermometer.

If you ever find yourself stuck or unsure about a negative number question, just draw yourself a very quick number line, count the spaces with your finger, and you will be fine.

I still do this, and I'm... well, quite a bit older then you.





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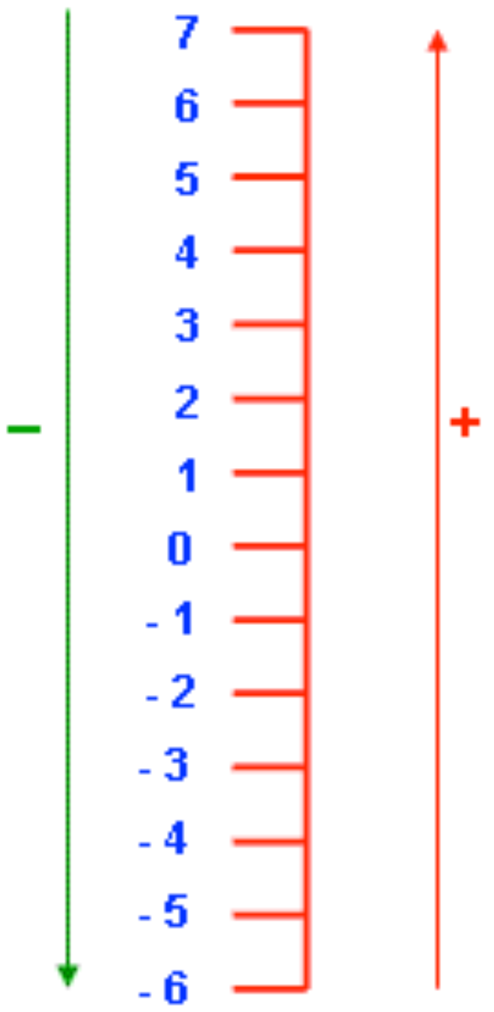
Adding and Subtracting when the Signs are NOT Touching

Where people seem to go wrong with negative numbers is that they learn the rule that two minuses make a plus.

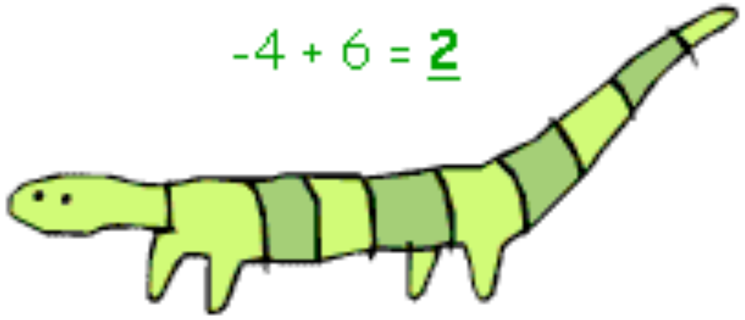
Now, this rule is a good one, but must only be used when two signs (+ or -) are touching.

If no signs are touching, I would just use this rule

Rule: If no signs are touching, use a number line (on paper or in your head), or think about money!



Example 1	**Example 2**	**Example 3**
2 - 7	-4 + 6	-1 - 4
Number Line: put your finger at 2 and move down 7 places	Number Line: put your finger at -4 and move up 6 places	Number Line: put your finger at -1 and move down 4 places
Money: if I have £2 in my bank and someone takes away £7, then how much do I have?	Money: if I have £-4 in my bank (I am in debt) and someone gives me £6, then how much do I have?	Money: if I have £-1 in my bank (I am in debt) and someone takes away £4, then how much do I have?
2 - 7 = **-5**	-4 + 6 = **2**	-1 - 4 = -**5**



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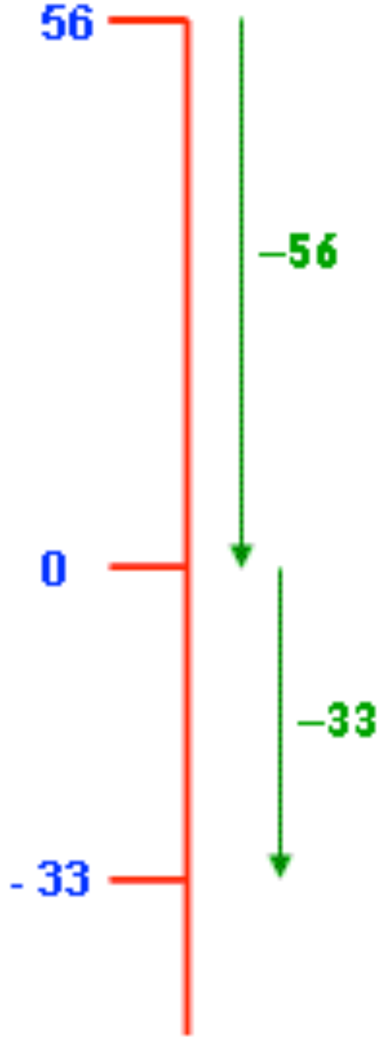
Now both these methods still work when the numbers become harder and the number line becomes too big to draw:

Example 4



56 - 89

Number Line: imagine your finger is as 56.

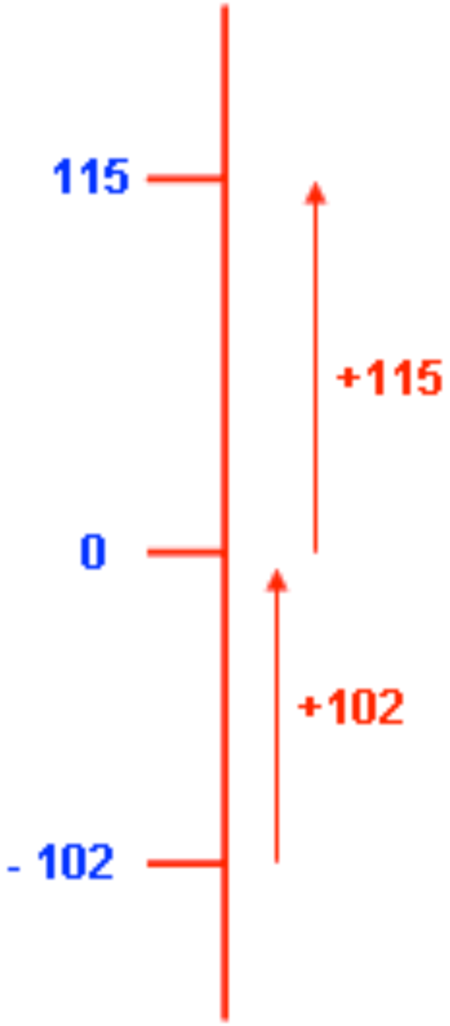
How far must you go down to get to zero?... 56 spaces, right?

And so, how much further down do you still have to go?... another 33 spaces!

Money: if I have £56 in my bank and someone takes away £89, then how much do I have?

56 - 89 = **-33**

Example 5



-102 + 217

Number Line: imagine your finger is as -102.

How far must you go up to get to zero?... 102 spaces, right?

And so, how much further up do you still have to go?... another 115 spaces!

Money: if I have £-102 in my bank and someone gives me £217, then how much do I have?

-102 + 217 = **115**

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Adding and Subtracting when the Signs ARE Touching

Okay, now it's time for our rule...

Rule: If two signs are touching (+'s or −'s next to each other), then replace the two signs with one sign using these rules:

+ and - = -

+ and + = +

- and + = -

- and - = +

Example 1

-4 + -8

Have you spotted the touching signs?...

Using our rule, we can change + and - to -

So, our sum becomes:

-4 - 8

Which is pretty easy using either number lines or money.

-4 + -8 = **-12**

Example 2

5 - -6

Have you spotted the touching signs?...

Using our rule, we can change - and - to +

So, our sum becomes:

5 + 6

Which is pretty easy however you do it!

5 - -6 = **11**

Example 3

-22 - - 9

Have you spotted the touching signs?...

Using our rule, we can change - and - to +

So, our sum becomes:

-22 + 9

Which is pretty easy using either number lines or money

-22 - -9 = -**13**

Example 4

-6 - 10

Have you spotted the touching signs?...

I hope not, because there aren't any!

The two minuses are NOT touching

So our sum stays the same and we do it using either of our methods

-6 - 10 = -**16**

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Multiplying and Dividing

As was the case with fractions, multiplying and dividing with negative numbers is a little bit easier than adding and subtracting, but you still have to concentrate!

Rule: Do the sum as normal, ignoring the plus and minus signs and write down the answer

Then, carefully count the number of minus signs in the question.

If there is one the whole answer is negative, if there are two the answer is positive, if there are three the answer is negative, four means positive, and so on...

Example 1

$-20 \div 4$

Do the sum as normal, ignoring the minus signs

$20 \div 4 = 5$

Count the number of minus signs in the question... **1**!

One minus makes the whole answer negative

So:

$-20 \div 4 = \underline{\mathbf{-5}}$

Example 2

-6×-9

Do the sum as normal, ignoring the minus signs

$6 \times 9 = 54$

Count the number of minus signs in the question... **2**!

Two minuses makes the whole answer positive

So:

$-6 \times -9 = \underline{\mathbf{54}}$

Example 3

$-3 \times -2 \times -5$

Do the sum as normal, ignoring the minus signs

$3 \times 2 \times 5 = 30$

Count the number of minus signs in the question... **3**!

Three minuses makes the whole answer negative

So:

$3 \times 2 \times 5 = \underline{\mathbf{-30}}$

Example 4

$^{-88}/_{-4}$

Do the sum as normal, ignoring the minus signs

$^{88}/_{4} = 22$

Count the number of minus signs in the question... **2**!

Two minuses makes the whole answer positive

So:

$^{-88}/_{-4} = 22$

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Tricky Questions involving Negative Numbers

The people who write maths exams are nasty. Just when you think you have got a topic sorted, they chuck in a right stinker.

But do not panic. So long as you remember the rules we have discussed here, and you don't forget old BODMAS/BIDMAS, you will be fine!

Example 1

$$-2\times(-3+5)$$

Now, BIDMAS says we must sort out the brackets first:

$$-3+5=2$$

So now we have:

$$-2\times2$$

And using our negative number rules, we should get the answer of:

= -4

Example 2

$$-3+-8\div4-2$$

Now, BIDMAS says we must sort out the division first:

$$-8\div4=-2$$

Putting that back in the question, we have:

$$-3+-2-2$$

Let's sort those two touching signs out:

$$-3-2-2$$

Using number lines, or money, we should get

= -7

Example 3

$$\frac{-4\times-3}{-3--9}$$

Now remember, even though we can't see any brackets, they are hidden on the top and bottom of the fraction:

$$\frac{(-4\times-3)}{(-3--9)}$$

So, the top gives us:

$$-4\times-3=12$$

And from the bottom:

$$-3--9$$

$$=-3+9=6$$

Leaving us with:

$$\frac{12}{6}$$

= 2

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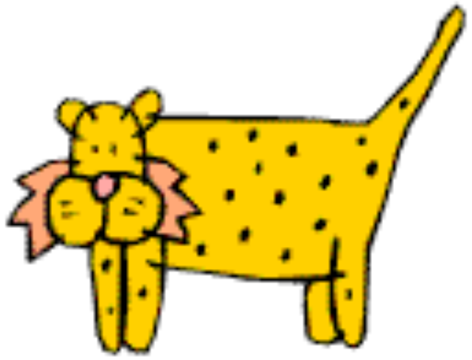
9. Sequences

Things you might need to be able to do with sequences...

This will vary with your age and what maths set you are in, but here is a list of some of the things you might need to be able to do with sequences:



1. Spot and describe number sequences
2. Work out the nth term of linear sequences
3. Write out terms of sequences given the rule
4. Work out the nth term of quadratic sequences

It's quite a nice little topic this one... well, as far as maths topics go, anyway!

What is a sequence?

A sequences is just a set of numbers which follows a rule.

The rule may be very simple, or very complicated, but the important thing is that every single number in that sequence follows the same rule.

The reason this is important is that allows you to predict what number will come next, and even what number will come in 1,000,000 numbers time!

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1. Spotting and Describing Number Sequences

In these types of questions you will usually be given a set of numbers, and be asked to describe the rule (how to you get from one number to the next), and predict what the next couple of numbers will be.

Let's have a look at some sequences together:

1.	7	10	13	16	19	...	...	
2.	3	6	12	24	48	...	...	
3.	200	190	181	173	166	...	...	
4.	1	1	2	3	5	8	13	...	...

---

1. Here the numbers are going up by 3 every time, so the rule is something like: "add 3 to the previous number to get the next number", and so the next two numbers are 22 and 25

2. Here the numbers are doubling (or multiplying by 2), so the rule is something like: "double the previous number to get the next number", and so the next two numbers are 96 and 192

3. This one is a little tricky to spot, and even trickier to describe. I would go for something like: "subtract 10 from the 1st number, 9 from the 2nd, 8 from the 3rd, and so on". So long as you are clear, you will get the marks. So, the next two numbers must be: 160 and 155

4. This is a sneaky one. It's a very famous sequence called "The Fibonacci Sequence". Here is the rule: "add the previous two numbers together to get the next number". That means what must come next are: 21 and 34

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2. Finding the nth term of Linear Sequences

Not the greatest sounding title in the world, hey?

Let's have a look at what each bit means, and you'll see it's not so bad:

nth term - well, term is just a posh word for the numbers in a sequence, and **n** is just the letter we use to describe the position of each term. So, n = 1 is the 1st term, and n = 5 is the 5th term. All "find the nth term" means is just to find a rule which allows us to work out what number lies at any position in our sequence.

linear sequences - these are just sequences where you either add or subtract the same number to get from term to term. For example, sequence 1. up above was a linear sequence because you added 3 each time, but number none of the other were (in 2. we multiplied, and in 3. and 4. we added or subtracted a different amount).

Now I have a method for finding the nth term of linear sequences:

1. Decide what you have to add or subtract to get from term to term (and make sure it is the same for each term!)

2. Write the times table of this number underneath the sequence (this gives you the number that goes in front of **n**)

3. Figure out what you have to do to your times table to get back to your sequence (this gives you the number at the end)

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Example 1

Find the nth term of the following sequence:

13 19 25 31 37 ... ...

1. Okay, now looking at the numbers I reckon you have to add 6 each time, but before I continue, I am just going to check each term to make sure… erm… yep, add 6 each time!

2. Adding 6 each time means two things: (1) 6 is the number in front of **n** in my rule, so I know there will be a **6n** involved (2) I need to write the 6 times table carefully under my sequence:

n	1	2	3	4	5	...	...
Sequence:	13	19	25	31	37	...	...
6n	6	12	18	24	30	...	...

Notice: I have also written **n** above the sequence, just to remind you that all **n** means is the position of the numbers in the sequence (n = 4 is the 4th number in the sequence, which is 31). Also, notice how the 6 times table is just 6 times as big as **n**!

3. Now you ask yourself: "what do I have to do to get from my 6 times table, back to my sequence?"… well, if you look carefully at the numbers, you will see you must… add 7 each time!

So, our rule for the nth term is… $6n + 7$

Which basically says that our sequence is just the 6 times table, with 7 added each time.

Notice: you can check you are correct by testing out your rule. We know the 5th term of the sequence is 37, but does are rule give us that?…

When $n = 5$ $\quad 6n + 7 \longrightarrow 6 \times 5 + 7 = \ldots 37!$ we are correct!

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Example 2

Find the nth term of the following sequence:

-20 -16 -12 -8 -4 … …

1. Okay, I know there are nasty negatives, but if you look carefully you should be able to see that you have to add 4 each time.

2. Again, adding 4 means two things: (1) 4 is the number in front of **n** in my rule, so I know there will be a **4n** involved (2) I need to write the 4 times table carefully under my sequence:

n	1	2	3	4	5	…	…
Sequence:	-20	-16	-12	-8	-4	…	…
4n	4	8	12	16	20	…	…

Notice: Again, I have put **n** on the top just to show you that that 4 times table is just 4 times as big as n!... that's all **4n** means, just get **n** and multiply it by 4!

3. Now you ask yourself: "what do I have to do to get from my 4 times table, back to my sequence?"… well, again you have to be careful, but I reckon you must… erm… subtract 24 each time!

So, our rule for the nth term is… $4n - 24$

Important: as well as checking, we can also use this rule to predict. For example, we can very quickly work out what the 100th term would be without writing out the whole sequence:

When $n = 100$ $4n - 24 \longrightarrow 4 \times 100 - 24 = \ldots 376!$

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Example 3

Find the nth term of the following sequence:

21 16 11 6 1 ... ...

1. Okay, this time looking at the numbers I reckon you have to subtract 5 each time.

2. Now, just because we are subtracting, our method still works! Subtracting 5 still means two things: (1) -5 is the number going in front of the **n** (2) we need a times table... the -5 times table!

n	1	2	3	4	5	...	...
Sequence:	21	16	11	6	1	...	...
-5n	-5	-10	-15	-20	-25	...	...

Notice: the -5 times table is just the 5 times table but with each term negative

3. Now you ask yourself: "what do I have to do to get from my -5 times table, back to my sequence?"... well, be very careful because of the negatives, but you must... add 26 each time!

So, our rule for the nth term is... $-5n + 26$

Important: Again, let's use our rule to predict. How about the 6th term:

When n = 6 $-5n + 26 \longrightarrow -5 \times 6 + 26 = \dots -4!$

Well, the 6th term was the next one in our sequence, and I reckon if we had worked it out in our head we would have got -4, so I think we are correct!

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3. Writing out the Terms of a Sequence given the Rule

Now, so long as you understand what n means, you will be fine with this!
Just to re-cap, **n** is just the position of the term in the sequence.
So... if you want the 5th term, then **n** must be 5!

Example 1

Write out the first 5 terms of the sequence whose nth term rule is: $7n - 3$

Okay, to get out 1st term, **n** must equal 1. So we have:

When $n = 1$ $7n - 3 \longrightarrow 7 \times 1 - 3 = \ldots \underline{4}$ so, 1st term is $\underline{4}$

Now to get out 2nd term, **n** must equal 2. So we have:

When $n = 2$ $7n - 3 \longrightarrow 7 \times 2 - 3 = \ldots \underline{11}$ so, 1st term is $\underline{11}$

And if you keep this going, you end up with the first 5 terms: 4 11 18 25 32

Notice: the gap between each term is +7... which is what we would have expected from the 7n!

Example 2

Write out the first 5 terms of the sequence whose nth term rule is: $n^2 + 10$

Looks hard, but same technique! To get out 1st term, **n** must equal 1. So we have:

When $n = 1$ $n^2 + 10 \longrightarrow 1^2 + 10 = \ldots \underline{11}$ so, 1st term is $\underline{11}$

Now to get out 2nd term, **n** must equal 2. So we have:

When $n = 2$ $n^2 + 10 \longrightarrow 2^2 + 10 = \ldots \underline{14}$ so, 1st term is $\underline{14}$

And if you keep this going, you end up with the first 5 terms: 1 14 19 26 35

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4. Work out the nth term of Quadratic Sequences

Now, not all sequences are nice little linear ones.

If you have a look at the last example, you will see that the terms do not go up by the same amount, and if you look at the nth term rule, you will see why… it's quadratic!

Now, there is a really complicated method for finding the nth term of quadratics, but 9 times out of 10, a much simpler method works, so long as you know your square numbers!

1. Write out the square numbers (n^2) underneath your sequence

2. Work out what you have to do to the square numbers to get back to your sequence

Example

Find the nth term of the following sequence:

-2 1 6 13 22 … …

1. The terms are NOT going up by the same amount each time, so we need the square numbers…

n	1	2	3	4	5	…	…
Sequence:	-2	1	6	13	22	…	…
n^2	1	4	9	16	25	…	…

2. What do you have to do to get from your square numbers back to your original sequence?… well, I reckon you need to… subtract 3!

So, our nth term rule is: $n^2 - 3$

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10. Surds

What on earth is a surd and why do we need them?...

Let's get it out of the way before we start... yes, I know, "surd" sounds a little bit like a rude word, and my pupils never tire of reminding me of that every lesson...

What are they?... Surds are just numbers left in square-root form, like $\sqrt{3}$ or $\sqrt{7}$

But why do we need them?... Because such numbers are irrational, and if we tried to write them out as decimals, they would go on forever!

The Two Important Rules of Surds

Everything we are going to look at in this section is based around these two crucial rules:

Rule 1

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

If you have a surd and you multiply it by another surd, then the answer is just the same as the surd of the original two numbers (a and b) multiplied together

e.g. $\sqrt{7} \times \sqrt{5} = \sqrt{7 \times 5} = \sqrt{35}$

Rule 2

$$\sqrt{a} \times \sqrt{a} = a$$

If you multiply a surd by itself, then the answer is just the original number before it was square-rooted

e.g.

$$\sqrt{8} \times \sqrt{8} = \sqrt{8 \times 8} = \sqrt{64} = 8$$

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1. Simplifying Single Surds

Okay, this is probably the nicest type of surd question you could get asked.
You need to make the number under the square root sign as small as possible
And it's nice and easy so long as you know your square numbers!

Method

1. Split up the number being square-rooted into a product of at least one square number
2. Use Rule 1 to simplify your answer

Remember: Square Numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100...

Example 1

Simplify: $\sqrt{50}$

Okay, so we need to split up 50. We ask ourselves: "which square number is a factor of 50?"

Well, if you look along the list above, you should notice that... 25 is!

$50 = 25 \times 2$ So, using Rule 1... $\sqrt{50} = \sqrt{25} \times \sqrt{2}$

Now, because we've chosen a square number, that's going to simplify nicely...

$\sqrt{25} = 5$ So... $\sqrt{50} = 5 \times \sqrt{2} = 5\sqrt{2}$

Example 2

Simplify: $\sqrt{45}$

Okay, so this time we need to split up 45. We ask ourselves: "which square number is a factor of 45?"

Well, if you look along the list above, you should notice that... 9 is!

$45 = 9 \times 5$ So, using Rule 1... $\sqrt{45} = \sqrt{9} \times \sqrt{5}$

Now, because we've chosen a square number, that's going to simplify nicely...

$\sqrt{9} = 3$ So... $\sqrt{45} = 3 \times \sqrt{5} = 3\sqrt{5}$

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2. Simplifying more than one Surd (Multiplying)

Again this is fairly easy so long as you could understand the previous section

Method

1. Deal with each surd individually
2. Split up the numbers being square-rooted into a product of at least one square number
3. Use Rule 1 to simplify your answers
4. When simplifying the whole answer, treat your whole numbers and surds separately

Remember: Square Numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100...

Example Simplify: $\sqrt{90} \times \sqrt{20}$

Okay, let's deal with each surd individually and split them up exactly like we did in the previous section:

$$90 = 9 \times 10 \longrightarrow \sqrt{90} = \sqrt{9} \times \sqrt{10} \longrightarrow \sqrt{90} = 3 \times \sqrt{10} = 3\sqrt{10}$$

$$20 = 4 \times 5 \longrightarrow \sqrt{20} = \sqrt{4} \times \sqrt{5} \longrightarrow \sqrt{20} = 2 \times \sqrt{5} = 2\sqrt{5}$$

So... $\sqrt{90} \times \sqrt{20} = 3\sqrt{10} \times 2\sqrt{5}$

To simplify further we multiply our whole number and our surds separately

$3 \times 2 = 6$ and... $\sqrt{10} \times \sqrt{5} = \sqrt{50}$ So... $3\sqrt{10} \times 2\sqrt{5} = 6\sqrt{50}$

And if you wanted to be really clever, we can simplify even further...

$\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$ So... $6\sqrt{50} = 6 \times 5\sqrt{2} = 30\sqrt{2}$

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3. Simplifying more than one Surd (Dividing)

Good News: Do these in exactly the same way as the Multiplying ones!

Example Simplify: $\dfrac{\sqrt{60}\times\sqrt{20}}{\sqrt{12}}$

Okay, let's dealwith each surd individually and split them up because we're good at that!...

$$60 = 4\times15 \longrightarrow \sqrt{60}=\sqrt{4}\times\sqrt{15} \longrightarrow \sqrt{60}=2\times\sqrt{15}=2\sqrt{15}$$

$$20 = 4\times5 \longrightarrow \sqrt{20}=\sqrt{4}\times\sqrt{5} \longrightarrow \sqrt{20}=2\times\sqrt{5}=2\sqrt{5}$$

$$12 = 4\times3 \longrightarrow \sqrt{12}=\sqrt{4}\times\sqrt{3} \longrightarrow \sqrt{12}=2\times\sqrt{3}=2\sqrt{3}$$

So...

$$\frac{\sqrt{60}\times\sqrt{20}}{\sqrt{12}}=\frac{2\sqrt{15}\times2\sqrt{5}}{2\sqrt{3}}$$

Let's sort out the multiplication on the top line like we did before...

$2\times2=4$ and... $\sqrt{15}\times\sqrt{5}=\sqrt{75}$ So... $2\sqrt{15}\times2\sqrt{5}=4\sqrt{75}$

But we can be clever again and go a wee bit further...

$\sqrt{75}=\sqrt{25}\times\sqrt{3}=5\sqrt{3}$ So... $4\sqrt{75}=4\times5\sqrt{3}=20\sqrt{3}$

So (after what seems like ages) we are left with:

$$\frac{\sqrt{60}\times\sqrt{20}}{\sqrt{12}}=\frac{20\sqrt{3}}{2\sqrt{3}}$$

But wait a minute! We can use division to simplify, just like we used multiplication...

$20\div2=10$ and... $\sqrt{3}\div\sqrt{3}=1$

So... $\dfrac{20\sqrt{3}}{2\sqrt{3}}=10\times1=10$

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4. Simplifying more than one Surd (Adding and Subtracting)

Just like when we are adding and subtracting fractions, there is a little twist!...

Twist

We can only add and subtract surds of the same type

So... we must use our simplifying skills to change them into the same type!

Example 1 Simplify: $\sqrt{12} + \sqrt{27}$

Now, one thing is for certain: the answer is definitely NOT: $\sqrt{39}$ No way! No such rule! Don't forget!

We need to simplify the surds to see if that helps

$$12 = 4 \times 3 \longrightarrow \sqrt{12} = \sqrt{4} \times \sqrt{3} \longrightarrow \sqrt{12} = 2 \times \sqrt{3} = 2\sqrt{3}$$

$$27 = 9 \times 3 \longrightarrow \sqrt{27} = \sqrt{9} \times \sqrt{3} \longrightarrow \sqrt{27} = 3 \times \sqrt{3} = 3\sqrt{3}$$

So...

$$\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3}$$

Look, our surds are now of the same type! They are both: $\sqrt{3}$

So we can now just add our whole numbers, because 2 lots of something, plus 3 lots of something must equal 5 lots of something! So we have our answer...

$$2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$$

Example 2 Simplify: $\sqrt{63} - \sqrt{28}$

Now, one thing is for certain: the answer is definitely NOT: $\sqrt{35}$ No way! No such rule! Don't forget!

We need to simplify the surds to see if that helps

$$63 = 9 \times 7 \longrightarrow \sqrt{63} = \sqrt{9} \times \sqrt{7} \longrightarrow \sqrt{63} = 3 \times \sqrt{7} = 3\sqrt{7}$$

$$28 = 4 \times 7 \longrightarrow \sqrt{28} = \sqrt{4} \times \sqrt{7} \longrightarrow \sqrt{28} = 2 \times \sqrt{7} = 2\sqrt{7}$$

So...

$$\sqrt{63} - \sqrt{28} = 3\sqrt{7} - 2\sqrt{7}$$

Look, our surds are now of the same type! They are both: $\sqrt{7}$

So we can now just subtract our whole numbers, because 3 lots of something, minus 2 lots of something must equal 1 lot of something! So we have our answer...

$$3\sqrt{7} - 2\sqrt{7} = \sqrt{7}$$

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5. Rationalising the Denominator!

Warning: This is hard, and should only be attempted by the very brave...

What does Rationalising the Denominator mean?...

Basically, it is considered a bit untidy in the fussy world of mathematics to have a surd on the bottom of a fraction (the denominator). So, if we can get rid of all the surds off the bottom of a fraction, we get rid of all the irrational numbers, and so we rationalise the denominator!

Method

Multiply the top and the bottom of the fraction by the same carefully chosen expression!

Example 1 – Single Surd

Rationalise the denominator of: $\frac{2}{\sqrt{3}}$

Okay, so we don't like the look of that $\sqrt{3}$ on the bottom

What could we multiply it by to make it disappear?... Well, using Rule 2... how about by itself!

Be careful: Remember, whatever we multiply the bottom of the fraction by, we must also do to the top, otherwise the value of the fraction changes, so we will have changed the question!

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

Using our Rules of Fractions, we just multiply the tops together, and then the bottoms together →

$$2 \times \sqrt{3} = 2\sqrt{3}$$

And using Rule 2...

$$\sqrt{3} \times \sqrt{3} = 3$$

So, we are left with our answer! $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

And if you check them on the calculator, you will see they give the same answer!

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Example 2 – Surd with Other Numbers

Rationalise the denominator of: $\frac{5}{3-\sqrt{2}}$

Trick

For questions like this, the thing you multiply both the top and the bottom of the fraction by is just the expression on the bottom, but with the sign changed!

Why, I hear you ask?... Well, it's all to do with the difference of two squares...

Okay, so let's multiply the top and the bottom of the fraction by... change the sign... $3+\sqrt{2}$

$$\frac{5}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

Again, we multiply tops and bottoms together, but we also use our methods of expanding brackets (see the Algebra section)

Tops $\longrightarrow$ $5\times(3+\sqrt{2})$ $\longrightarrow$ $15+5\sqrt{2}$

Bottoms   Use FOIL $\longrightarrow$ $(3-\sqrt{2})\times(3+\sqrt{2})$ $\longrightarrow$ $9+3\sqrt{2}-3\sqrt{2}-2$

Now look what happens when we collect up our terms and simplify $\longrightarrow$ $9-2=7$

The middle two terms cancel out, and we are left with a very nice (and rationalised) denominator!

So... our answer must be...

$$\frac{5}{3-\sqrt{2}} = \frac{15+5\sqrt{2}}{7}$$

And if you check them on the calculator, you will see they give the same answer!

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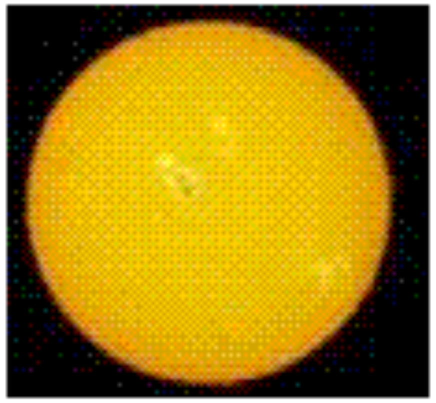
11. Standard Form



What is Standard Form and why do we need it?...

How heavy do you reckon the sun is?...

I'll tell you, its about: 2 000 000 000 000 000 000 000 000 000 000 kg

Now, I don't know about you, but I can't be bothered either counting or writing out all those zeros... well, fear not, because that is why we have standard form!

Standard form is just a convenient way of writing out really big or really small numbers.

Something really big like 2 000 000 000 000 000 000 000 000 000 000 kg is written as:

$$2 \times 10^{30} \text{ kg}$$

And something really small like: 0.00000000000000022 seconds is written as

$$2.2 \times 10^{-17} \text{ seconds}$$

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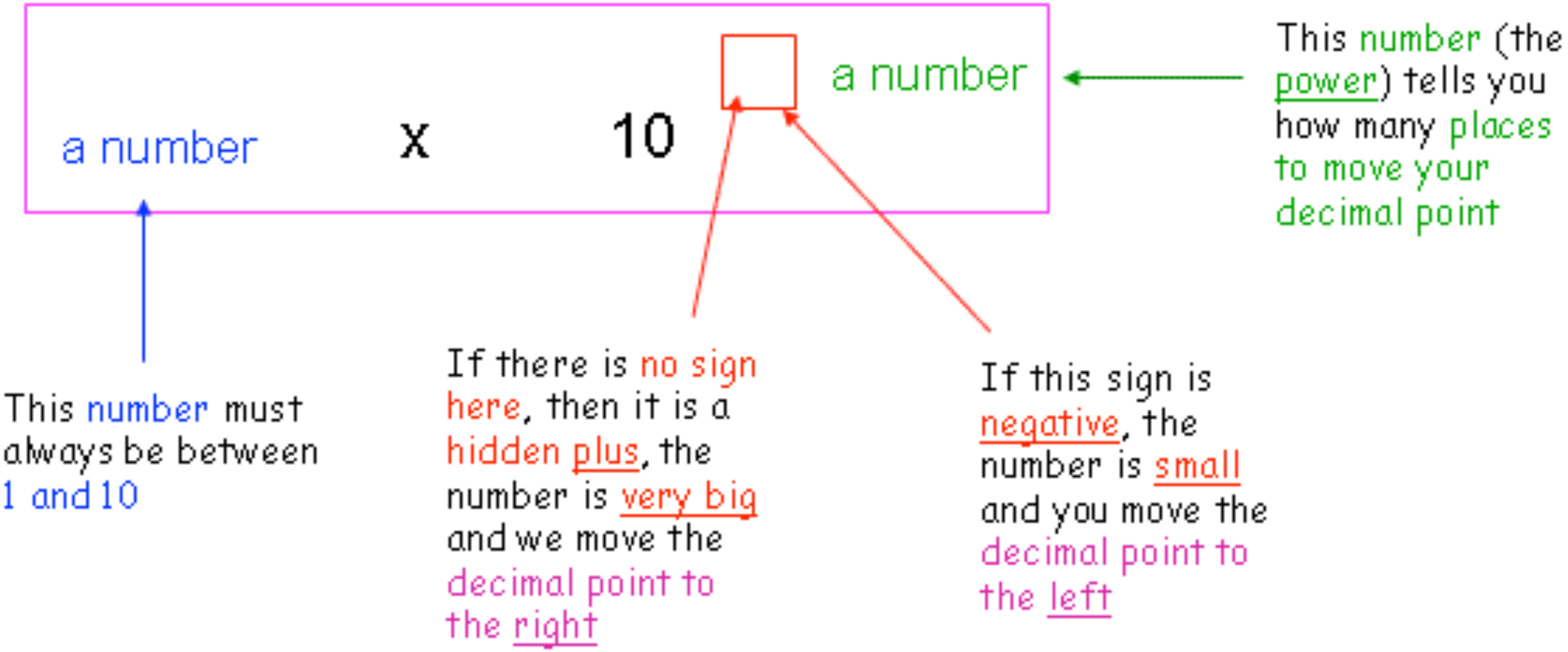
The Big Facts about Standard Form

When a number is written in standard form, it looks like this:



This number (the power) tells you how many places to move your decimal point

This number must always be between 1 and 10

If there is no sign here, then it is a hidden plus, the number is very big and we move the decimal point to the right

If this sign is negative, the number is small and you move the decimal point to the left

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1. Writing Numbers in Standard Form

Method

1. Place you finger where the decimal point is (it may be hidden!)

2. Count backwards or forwards the number of places you have to move to make the starting number between 1 and 10

3. Write your answer in standard form

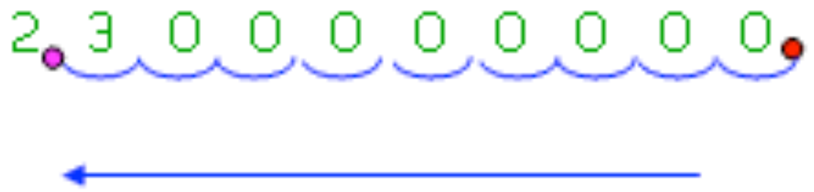
Example 1 2300000000

Now, with whole numbers like this, the decimal point is hidden at the end:

2 3 0 0 0 0 0 0 0 0.

Now, all we need to do is count how many places we need to move the decimal point until we create a number between 1 and 10

Well, I reckon the number we want is 2.3...



We have moved the decimal point 9 places, so our answer is...

$$2.3 \times 10^9$$

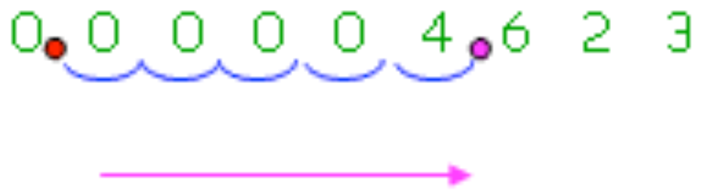
Example 2 0.00004623

Now, with decimals like this we can see the decimal point quite clearly!

0.0 0 0 0 4 6 2 3

Now, all we need to do is count how many places we need to move the decimal point until we create a number between 1 and 10

Well, I reckon the number we want is 4.623...



We have moved the decimal point 5 places, so our answer is...

$$4.623 \times 10^{-5}$$

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2. Changing From Standard Form

Method

Same thing as before, but this time you kind of need to work backwards.

Crucial: It is so easy to check your answer and so easy to make a mistake, so **check**!

Example 1 1.02×10^6

Okay, so we can see where the decimal point is, and the **6** flying in the air says we must move it 6 places to the right!

So, it looks like our answer is...

1 0 2 0 0 0 0

But don't take my word for it. Do what we did in the last section, and use your finger to work back from the answer

If you start with 1 0 2 0 0 0 0 and move your finger back 6 places, do you end up with... 1.02×10^6

Yes, so you've definitely got it right!

Example 2 7.6×10^{-5}

Okay, so we can see where the decimal point is, and the **-5** flying in the air says we must move it 5 places to the left!

Just like with the previous example, fill in the gaps with zeros...

So, it looks like our answer is...

0 0 0 0 0 7 6

Again, its so easy to check, so do it!

If you start with 0 0 0 0 0 7 6 and move your finger 5 places, do you end up with...

7.6×10^{-5}

Yes, so you've definitely got it right!

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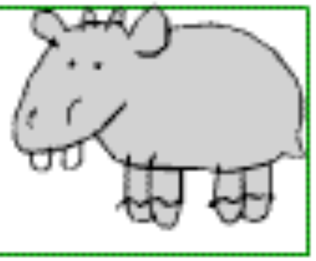
3. Multiplying and Dividing with Standard Form

Method

This is actually quite nice. All you need to do is...

Multiply/Divide your big numbers, Add/Subtract your powers



Example 1 $(8\times10^7)\times(5\times10^2)$

Okay, let's follow our method:

Multiply our Big Numbers:

$$8\times5=40$$

Add our Powers...

$$10^7\times10^2=10^9$$

So, it looks like our answer is...

$$40\times10^9$$

Problem: This answer is NOT in Standard Form, because 40 is not between 1 and 10

So we must use our brains to change it...

$$40\times10^9=4\times10^{10}$$

Our extra zero... goes here!

Example 2 $\dfrac{3\times10^5}{5\times10^2}$

Okay, let's follow our method:

Divide our Big Numbers:

$$3\div5=0.6$$

Subtract our Powers...

$$10^5\div10^2=10^3$$

So, it looks like our answer is...

$$0.6\times10^3$$

Problem: This answer is NOT in Standard Form, because 0.6 is not between 1 and 10

So we must use our brains to change it...

$$0.6\times10^3=6\times10^2$$

We need to borrow a zero... from here!

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4. Adding and Subtracting with Standard Form

Method

Unfortunately, there is no easier way to do this than...

Write out the numbers in full and then add or subtract the old fashioned way!

Example 1 $(2.3\times10^{4}) + (4.31\times10^{5})$

Okay, so first we must change both numbers into standard form:

$(2.3\times10^{4}) \longrightarrow 23000$

$(4.31\times10^{5}) \longrightarrow 431000$

Now we line our digits up carefully and add...

$$\begin{array}{r} 431000 \\ +\quad 23000 \\ \hline 454000 \end{array}$$

Usually you will then be asked to convert your answer back into Standard Form...

$454000 = 4.54 \times 10^{5}$

Example 2 $(8.32\times10^{-3}) - (3.8\times10^{-4})$

Okay, so first we must change both numbers into standard form:

$(8.32\times10^{-3}) \longrightarrow 0.00832$

$(3.8\times10^{-4}) \longrightarrow 0.00038$

Now we line our digits up carefully and subtract...

$$\begin{array}{r} 0.00832 \\ -\quad 0.00038 \\ \hline 0.00794 \end{array}$$

Usually you will then be asked to convert your answer back into Standard Form...

$0.00794 = 7.94 \times 10^{-3}$

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12. Ratio

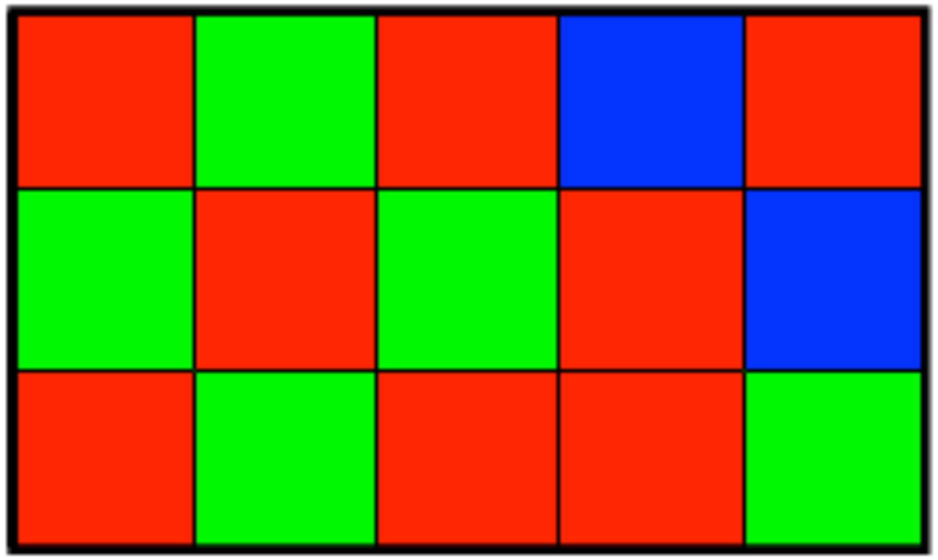
What are Ratios and Why do we need them?...

Ratios are just a nice easy way of showing the relative sizes of something, whether it be quantities of money, lengths of desks, amount of time, pretty much anything you can measure can be expressed as a ratio.

Ratios are also very closely linked to Fractions, and they behave in a very similar way. So... if you can understand fractions, you'll be flying here!

1. Writing Ratios

There is a funny way of ratios that requires the use of a colon **:** let me show you...



The ratio of red squares to green squares is:

8 : 5

Because for every 8 red squares, there are 5 green:

The ratio of green squares to red squares is:

5 : 8

The ratio of blue squares to red squares is:

2 : 8

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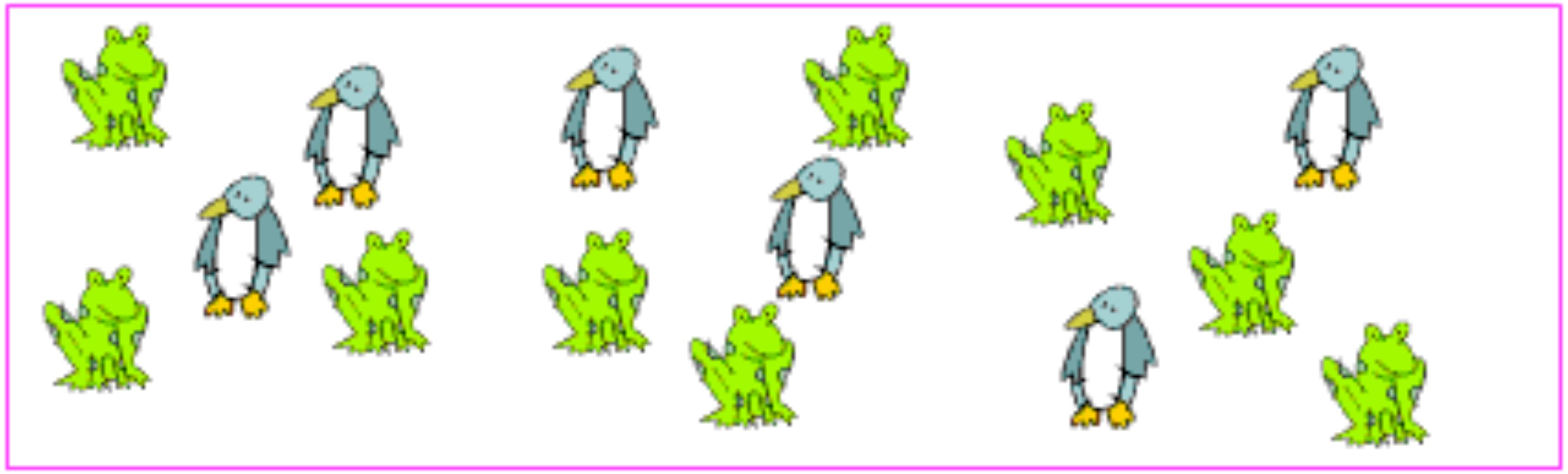
2. Simplifying Ratios

For the whole box, the ratio of frogs to penguins is:

9 : 6

But, can you see that for every 3 frogs, there are 2 penguins?... So, it's also:

3 : 2



Method for Simplifying Ratios

Just like with fractions, whatever you multiply/divide one side by, make sure you do the exact same to the other side!

Keep dividing until each side has no common factors

Example 1

Simplify 14 : 21

Okay, we are looking for factors common to both sides... how about **7**!

Divide both sides by 7...

14 : 21

÷7 ÷7

2 : 3

Check for other common factors to make it even simpler?... No, so we're done!

Example 2

Simplify 60 : 45

Okay, we are looking for factors common to both sides... how about **15**!

Divide both sides by 15...

60 : 45

÷15 ÷15

4 : 3

Check for other common factors to make it even simpler?... No, so we're done!

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3. “1 to n” and “n to 1”!

Sometimes the mean examiners aren't happy with you merely simplifying a ratio, they want it expressed as either 1 : n or n : 1.

Sounds hard, but so long as you can simplify ratios, and you remember that **n** is just a number, you'll be fine!

Example 1 Express 8 : 14 in the form 1 : n

Right, what this question is asking you to do is to change 8 : 14 into 1 : n, where n is just a number for you to find.

Now, the important thing here is that you stick to the rule: whatever you multiply/divide one side by, do the exact same to the other side.

We need to change the 8 into 1, so we must divide by… erm… **8**!


8 : 14

÷8 ÷8

1 : ?

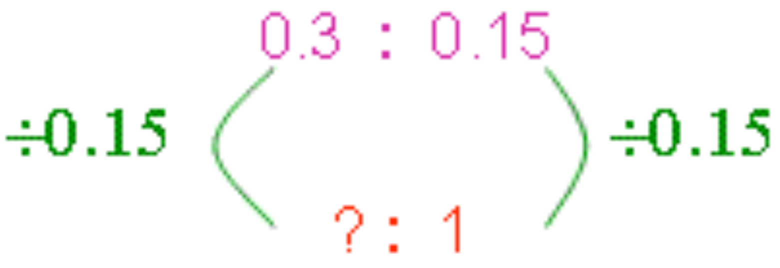
Dividing our other side by 8 gives us our final answer…

1 : 1.75

Example 1 Express 0.3 : 0.15 in the form n : 1

Again, we just need to change 0.3 : 0.15 into n : 1, sticking to our rule.

Problem: what on earth do we divide 0.15 by to give us 1?... Well, anything divided by itself is 1, so how about by 0.15!


0.3 : 0.15

÷0.15 ÷0.15

? : 1

So, to get the other side, we just divide 0.3 by 0.15, which gives us our answer…

2 : 1

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4. Classic Ratio Questions

The types of questions on ratio that you usually get in the exam sound really nasty, but all they require is a little knowledge of what we have done before.

Remember: Whatever you multiply/divide one side by, do the same to the other!

Example

Mr Barton has conned his Mum into making him a cake. It says on the packet that the ingredients must be mixed in the following ratios:

Flour (g)	:	Butter (g)	:	Eggs	:	Sugar (g)
400	:	220	:	3	:	25

(a) If my Mum has 1000g of flour, how much butter does she need?

(b) If she has 2 eggs, how much sugar does she need?

Always set these sort of questions out the same way - write the original ratios on the top, write the new amount you know on the bottom, and ask yourself: "what do I need to do to get from my original amount to my new amount?"

(a) This is what we've got:

flour butter

400 : 220

$\times 2.5$ $\times 2.5$

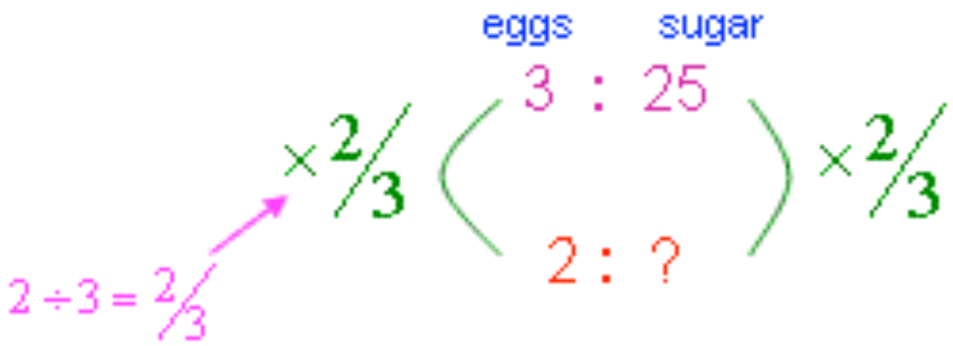
1000 : ?

$1000 \div 400 = 2.5$

How do I get from 400 to 1000?... I multiply by 2.5!, so let's do the same to the butter!

$220 \times 2.5 = \underline{550g}$

(b) This is what we've got:



eggs sugar

3 : 25

$\times \frac{2}{3}$ $\times \frac{2}{3}$

2 : ?

$2 \div 3 = \frac{2}{3}$

How do I get from 3 to 2?... I multiply by $\frac{2}{3}$ so let's do the same to the sugar!

$25 \times \frac{2}{3} = 16\frac{2}{3}g$

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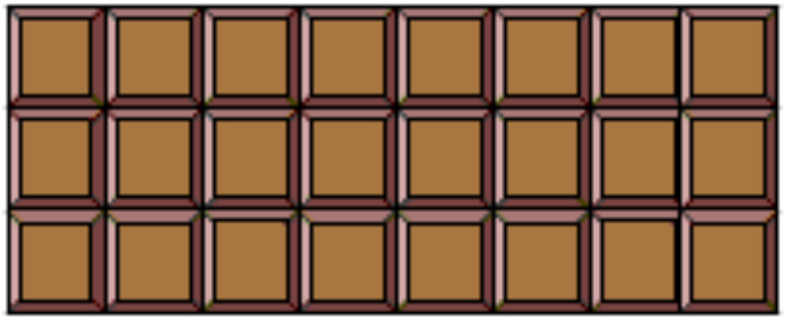
5. Sharing in a Given Ratio

For baking me the cake, I decide to share this bar of chocolate with my Mum in the ratio 5 : 3 (I am the 5, of course). How many pieces does each of us get?



Method for Sharing Ratios

1. Add up the total number of parts you are sharing between

2. Work out how much one part gets

3. Use this to work out how much everybody gets!

Example 1

The Chocolate Example!

1. Okay, so I get 5 parts, and my Mum gets 3 parts, so in total there are… 8 parts!

2. There are 24 pieces of chocolate all together, so each part must be worth…

24 ÷ 8 = 3 pieces

3. I have 5 parts, so I get:

3 x 5 = 15 pieces

And Mum's 3 parts get her:

3 x 3 = 9 pieces

Look: 15 + 9 = 24!

Example 2

Share £845 in the ratio 8 : 3 : 2

1. Okay, so in total there are… 13 parts!

2. We have £845 to share, so each part receives… 845 ÷ 13 = £65

3. How much does each person get?…

8 parts 65 x 8 = £520

3 parts 65 x 3 = £195

2 parts 65 x 2 = £130

Look: 520 + 195 + 130 = £845!

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13. Proportion

∝

What does proportion mean, and what's that funny fish symbol?

If two variables are proportional to each other, it just means that they are related to each other in a specific way.

The funny fish symbol ∝ just means "is proportional to"

1. Two types of Proportion

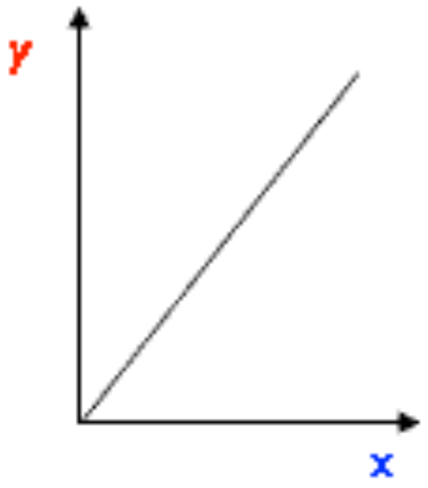
Again, how many of these you need to worry about depends on your maths set, and your exam board, and stuff like that, but here are the two main types of proportion:

(a) Direct Proportion

Both variables increase or decrease together

(i) Linear

Graph



Fancy Lingo

$$y \propto x$$

y is proportional to x

y is directly proportional to x

y varies directly as x varies

Example

x could be the number of KitKat Chunkys that you buy

y could be the total cost of those KitKat Chunkys

As the number you buy increases, so too does the total cost

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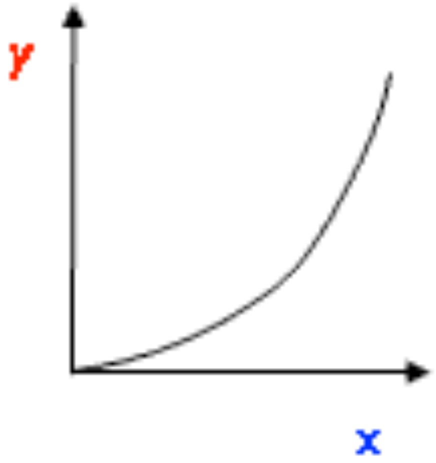
(ii) Quadratic

Graph



Fancy Lingo

$$y \propto x^2$$

y is proportional to x^2

y is directly proportional to x^2

y varies directly as x^2 varies

Example

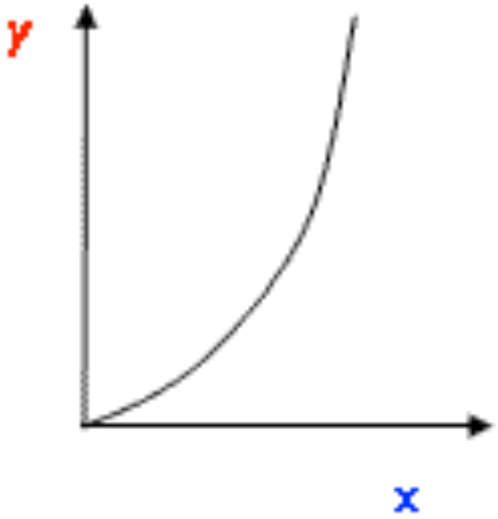
x could be the amount of money you spend advertising a gig

y could be the number of people who turn up to the gig

As the amount of advertising increases, word of mouth quickly spreads, and the number of people who go to the gig goes up by a lot.

(iii) Cubic

Graph



Fancy Lingo

$$y \propto x^3$$

y is proportional to x^3

y is directly proportional to x^3

y varies directly as x^3 varies

Example

x could be the amount of time you spend on mrbartonmaths.com

y could be your maths exam mark

As the amount of time you spend revising on the site increases, everything begins to fall into place, and your marks just get higher and higher with each extra minute!

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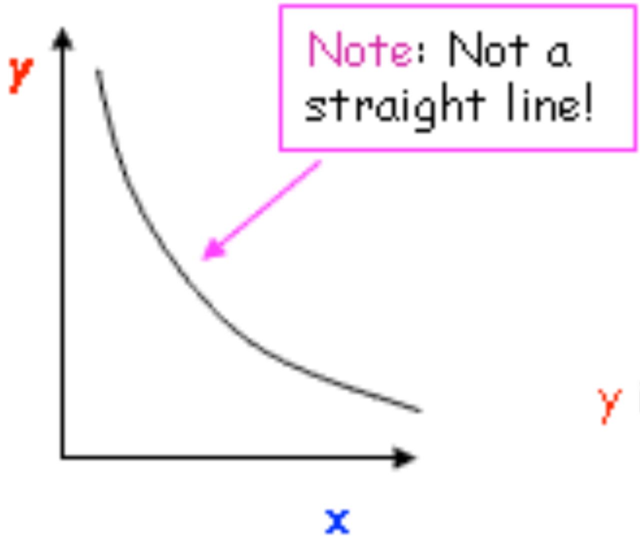
(b) Inverse Proportion

As one variable goes up, the other goes down

(i) Inverse

Graph



Note: Not a straight line!

Fancy Lingo

$$y \propto 1/x$$

y is inversely proportional to x

Example

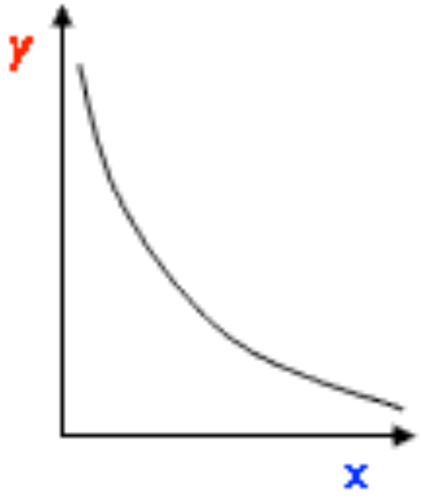
x could be the number of people you convince to join you on a road trip

y could be the amount each person must pay for petrol

As the number of people in the car increases, the amount everyone has to pay falls

(ii) Quadratic Inverse

Graph



Fancy Lingo

$$y \propto 1/x^2$$

y is inversely proportional to x^2

Example

x could be the number of hours you spend watching Big Brother

y could be your number of brain cells

As the hours increase, your brain cells disappear and an increasing rate!

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2. How to tackle proportion questions

Whatever the question, whatever the type of proportion, this method will never let you down… hopefully!

Method

1. Decide on the type of Proportion
   - Direct or Indirect?
   - Linear, Quadratic or Cubic?
2. Write the expression with the funny fish sign
3. Make the expression into an equation by using = and **k**
4. Use the numbers they give you to find out the value of **k**
5. Write down the formula
6. Answer the questions!

And now I'll take you through some pretty nasty examples, but each time we will use the same method, and everything will be fine… I promise!

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Example 1

y is directly proportional to x. Given that y = 12 when x = 4, calculate the value of:
(a) y when x = 6 (b) x when y = 66

1. Okay, we're in luck! The question has told us that we are dealing with direct proportion, and unless it says otherwise, we can also assume that it is linear.

2. The expression to say that y is directly proportional to x is: $y \propto x$

3. Okay, this is the key bit. As I said at the start, proportional means related to in a specific way. Indeed, once you decide what kind of proportion you are dealing with, all you need to do to get from x to y is to multiply by a number, which we call **k**.

Rule: Replace the $\propto$ sign with **=** and multiply the right hand side by **k**

$$y \propto x \longrightarrow y = kx$$

4. Now we use the numbers in the question and put them in our formula… when y = 12, x = 4

$$12 = 4k$$

And a little bit of rearranging gives us the value of **k** $k = {}^{12}\!/_{4} = 3$

5. So now we have our formula: $y = 3x$

6. And now life is easy because we just need to substitute in numbers and maybe (for b) rearrange!!

(a) Find y when x = 6

$$y = 3x \longrightarrow y = 3 \times 6$$

So… $y = 18$

(b) Find x when y = 66

$$y = 3x \longrightarrow x = {}^{y}\!/_{3} \longrightarrow x = {}^{66}\!/_{3}$$

So… $x = 22$

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Example 2

The variables p and q are related so that p is directly proportional to the square of q. Complete the table of values

p	0.5	2	
q		12	27

1. Again, we're in luck! The question has told us that we are dealing with direct proportion, and it also mentions the word "square" which means we are dealing with quadratic.

2. The expression to say that p is directly proportional to the square of q is: $p \propto q^2$

3. Rule: Replace the $\propto$ sign with **=** and multiply the right hand side by **k**

$$p \propto q^2 \longrightarrow p = kq^2$$

4. Now we look at the table and see what it tells us…. Well, it looks like when p = 2, q = 12, so let's put that information into our formula!

$$2 = 12^2 k \longrightarrow 2 = 144k$$

And a little bit of rearranging gives us the value of **k** $k = 2/144 = 1/72$

Note: I prefer a fraction to the horrible decimal

5. So now we have our formula: $p = \frac{1}{72}q^2$

6. And now life is easy because we just need to substitute in numbers and maybe (for b) rearrange!!

(a) Find p when q = 27

$$p = \frac{1}{72}q^2 \longrightarrow p = \frac{1}{72}27^2 \longrightarrow p = \frac{1}{72}\times 729$$

So… $p = 10.125$

(b) Find q when p = 0.5

$$p = \frac{1}{72}q^2 \longrightarrow q^2 = 72p \longrightarrow q^2 = 72\times 0.5 \longrightarrow q^2 = 36$$

So… $q = 6$

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Example 3

z is inversely proportional to t. Given that when t = 0.3 the value of z = 16, find the value of z when t = 0.5

1. Happy days! The question has told us that we are dealing with inverse proportion, and unless it says otherwise, we can also assume that there are no nasty squares or cubes around.

2. The expression to say that z is inversely proportional to t is: $z \propto \dfrac{1}{t}$

3. Rule: Replace the $\propto$ sign with **=** and multiply the right hand side by **k**, but be careful here!

$$z \propto \frac{1}{t} \longrightarrow z = \frac{k}{t}$$

Note: We are multiplying by k, so it goes on the top!

4. Now we use the numbers in the question and put them in our formula... when z = 16, t = 0.3

$$16 = \frac{k}{0.3}$$

And a little bit of rearranging gives us the value of **k** $\quad k = 0.3 \times 16 = 4.8$

5. So now we have our formula: $z = \dfrac{4.8}{t}$

6. And now life is easy because we just need to substitute in numbers:

(a) Find z when t = 0.5

$$z = \frac{4.8}{t} \longrightarrow z = \frac{4.8}{0.5} \longrightarrow$$ **So...** $z = 9.6$

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Example 4 - Nightmare!!!

(a) Describe the variation using $\propto$
(b) Find the equation connecting the two variables

y	5	125
x	5	25

1. Okay, we have a problem! The question doesn't tell us what type of proportion we are dealing with… but wait a minute, as x goes up (from 5 to 25), so does y (from 5 to 125), so it must be Direct.

Now, look at this…

x 25 or x 5^2

y	5	125
x	5	25

x 5

But what type of direct proportion?...

Well, to get from the first x value (5) to the second (25), you must multiply by 5. But for the y values, you must multiply by 25. Now, 25 is 5^2, so we must be talking quadratic proportion

(a) The expression to say that y is directly proportional to the square of x is: $y \propto x^2$

3. For the equation, we do what we always do… Rule: Replace the $\propto$ sign with **=** and multiply the right hand side by **k**

$$y \propto x^2 \longrightarrow y = kx^2$$

4. Now we have two sets of values to choose from to find k! I am going to choose the second, but it doesn't matter, the answer will be the same… when y = 125, x = 25

$$y = kx^2 \longrightarrow 125 = k25^2 \longrightarrow 125 = k625 \longrightarrow k = \frac{125}{625} \longrightarrow k = 0.2$$

(b) So… the equation is: $y = 0.2x^2$

Which works with both sets of values in the table!

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Algebra

1. The Rules of Algebra

Let's just get one thing clear before we start...

Algebra really isn't anything to be afraid of, I promise

If anything, dealing with letters is a lot easier than just dealing with numbers.

Why?... because, as you will see, letters are always cancelling each other out, meaning the questions get easier and easier the more you get into them,

And... quite often you can know for sure if your answer is correct, or not!

So, take a deep breath, think positive thoughts, and let's give this Algebra thing a go...

What is Algebra and Why do we need it?

- On a simple level, Algebra is just maths with letters... but it is a lot more than that!
- By bringing in letters as well as numbers we can work out things that numbers alone would not allow us to.
- In Algebra, letters are called "Unknowns". Basically, we stick a letter in to stand for something when we don't know it's true value.
- Now, this could be anything from the price of Nintendo Wii, the number of hours you spend watching TV in a week, or the speed you walk to school in the morning.
- If we don't know what it is, call it a letter – any letter you like – and let's let algebra figure everything out for us.

And the whole of Algebra – right up to A Level and beyond – is built around 3 rules...

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The Lingo You Need:

Term – this is basically any part of an expression or equation that involves a letter

e.g. 4m -2r and p are all terms

Expression – this is kind of like a collection of terms, maybe with a few numbers chucked in e.g. 4m + 2r and $8z - 5p + 6q^2 - 7$ are all expressions

Equation – this is just the same as an expression, but with an equals sign

e.g. 4m + 2r = 7 and 8z – 5p + 6q2 – 7 = a

Rule 1: You can add or subtract LIKE TERMS but you cannot add or subtract DIFFERENT TERMS.

Okay, so by a LIKE TERM I mean a term that contains the exact same letter (or letters) as another term

e.g. $m + m = 2m$ $3p + 2p = 5p$ $16t^2 - 4t^2 = 12t^2$ $10pq - 7pq = 3pq$

3 lots of something, plus 2 lots of something, gives you 5 lots of something

16 lots of something, minus 4 lots of something, gives you 12 lots of something

BUT...

$m + p$ Does Not = mp $3r + 2t$ Does Not = $5rt$

Because the terms are different!

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Simplifying Expressions

Now, once you have got to grips with Rule 1, it allows you to simplify nasty looking expressions into nice simple ones... which is called, believe it or not... simplifying.

To Simplify and Expression: Draw boxes around all the LIKE TERMS and deal with each set of like terms on their own.

Example 1

Simplify: $4m + 2p - m + 6p$

Okay, let's draw boxes around all the LIKE TERMS

Remember: Draw around the sign in front on the term as well!

$$\boxed{4m}\boxed{+2p}\boxed{-m}\boxed{+6p}$$

So, let's see what we've got:

🟥 $4m - m = 3m$

🟦 $2p + 6p = 8p$

Which gives us our answer of: $3m + 8p$

Note: if you cannot see a sign in front of a term, then just assume it is a PLUS

Example 2 - Tricky!

Simplify: $4t^2 - 5t - 2t - 3t^2$

Okay, let's draw boxes around all the LIKE TERMS

Remember: t and t² are DIFFERENT!

$$\boxed{4t^2}\boxed{-5t}\boxed{-2t}\boxed{-3t^2}$$

So, let's see what we've got:

🟥 $4t^2 - 3t^2 = t^2$

Note: write this instead of $1t^2$

🟦 $-5t - 2t = -7t$

Note: see how important it is you remember how to work with NEGATIVE NUMBERS!

Which gives us our answer of: $t^2 - 7t$

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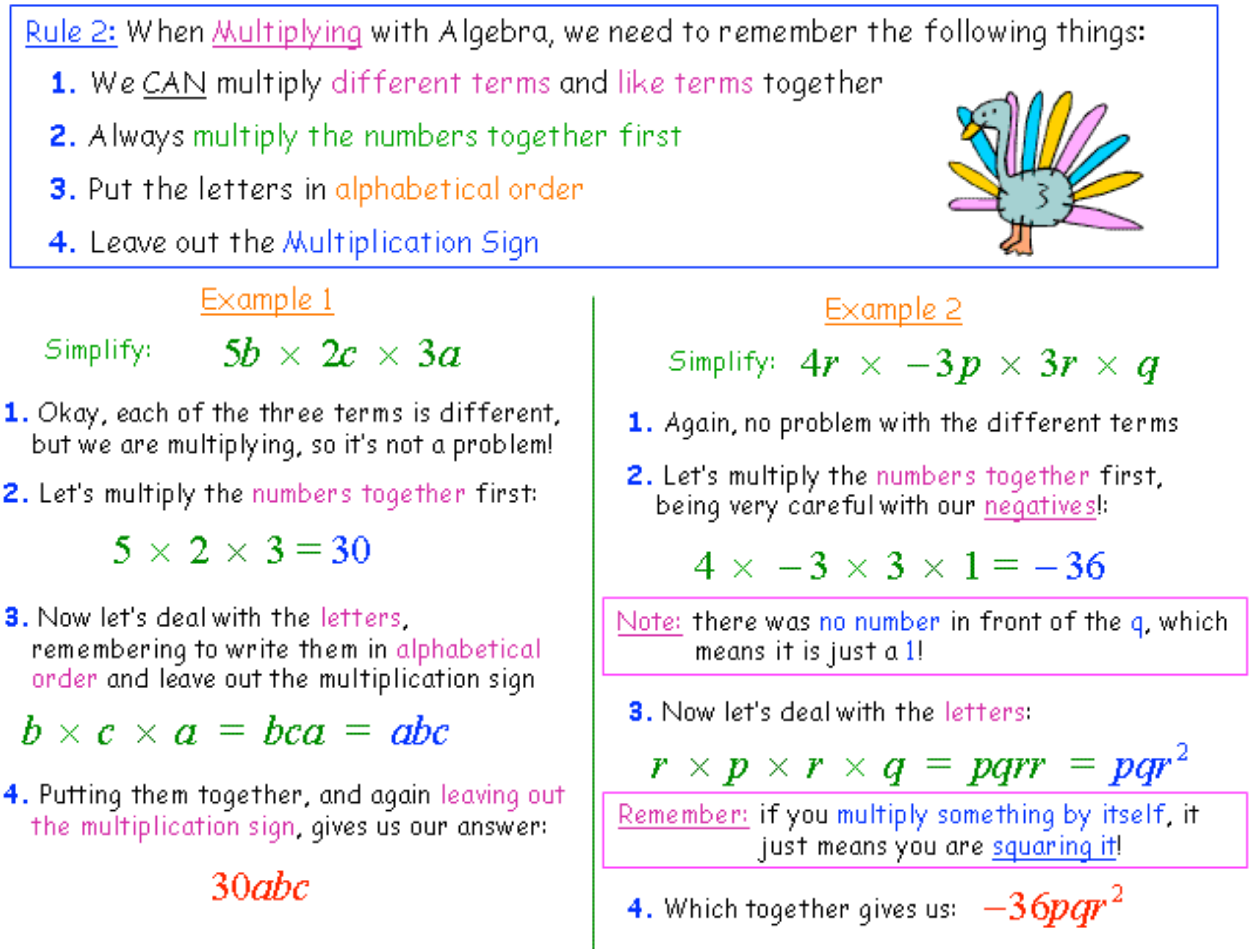

Rule 2: When Multiplying with Algebra, we need to remember the following things:

1. We CAN multiply different terms and like terms together
2. Always multiply the numbers together first
3. Put the letters in alphabetical order
4. Leave out the Multiplication Sign

Example 1

Simplify: $5b \times 2c \times 3a$

1. Okay, each of the three terms is different, but we are multiplying, so it's not a problem!
2. Let's multiply the numbers together first:

$$5 \times 2 \times 3 = 30$$

3. Now let's deal with the letters, remembering to write them in alphabetical order and leave out the multiplication sign

$$b \times c \times a = bca = abc$$

4. Putting them together, and again leaving out the multiplication sign, gives us our answer:

$$30abc$$

Example 2

Simplify: $4r \times -3p \times 3r \times q$

1. Again, no problem with the different terms
2. Let's multiply the numbers together first, being very careful with our negatives!:

$$4 \times -3 \times 3 \times 1 = -36$$

Note: there was no number in front of the q, which means it is just a 1!

3. Now let's deal with the letters:

$$r \times p \times r \times q = pqrr = pqr^2$$

Remember: if you multiply something by itself, it just means you are squaring it!

4. Which together gives us: $-36pqr^2$

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Rule 3: When Dividing with Algebra, the rules are just the same as when multiplying, but instead of a division sign like this ÷ we tend to write divisions as fractions!

Crucial: When dividing, watch for things cancelling out and disappearing!

Example 1

Simplify: $\dfrac{20xyz}{4z}$

1. Okay, just like when multiplying, different terms are no problem!

2. Let's divide the numbers first:

$$20 \div 4 = 5$$

3. Now let's deal with the letters:

$$xyz \div z = xy$$

What happened there? well, when you divide the z on the top by the z on the bottom you are left with **1** (just like if you divide anything by itself you get 1). But multiplying or dividing by 1 does not make a difference to our answer, so we can say that the z cancelled out!

4. So, our answer is: $5xy$

Example 2 - Nightmare!

Simplify: $\dfrac{5a^2b}{35ab^3}$

1. Different terms, no problem.

2. Dividing the numbers first:

$$5 \div 35 = \frac{5}{35} = \frac{1}{7}$$

Note: when you don't get a nice answer like in Example 1, you need to use Fractions!

3. Now let's deal with the letters (this requires a bit of knowledge about INDICES):

the a on the bottom wipes out one a on top, but still leaves an a behind on the top

the b^3 on the bottom wipes out the b on the top, and still leaves a b^2 behind on the bottom.

4. So, our answer is: $\dfrac{a}{7b^2}$

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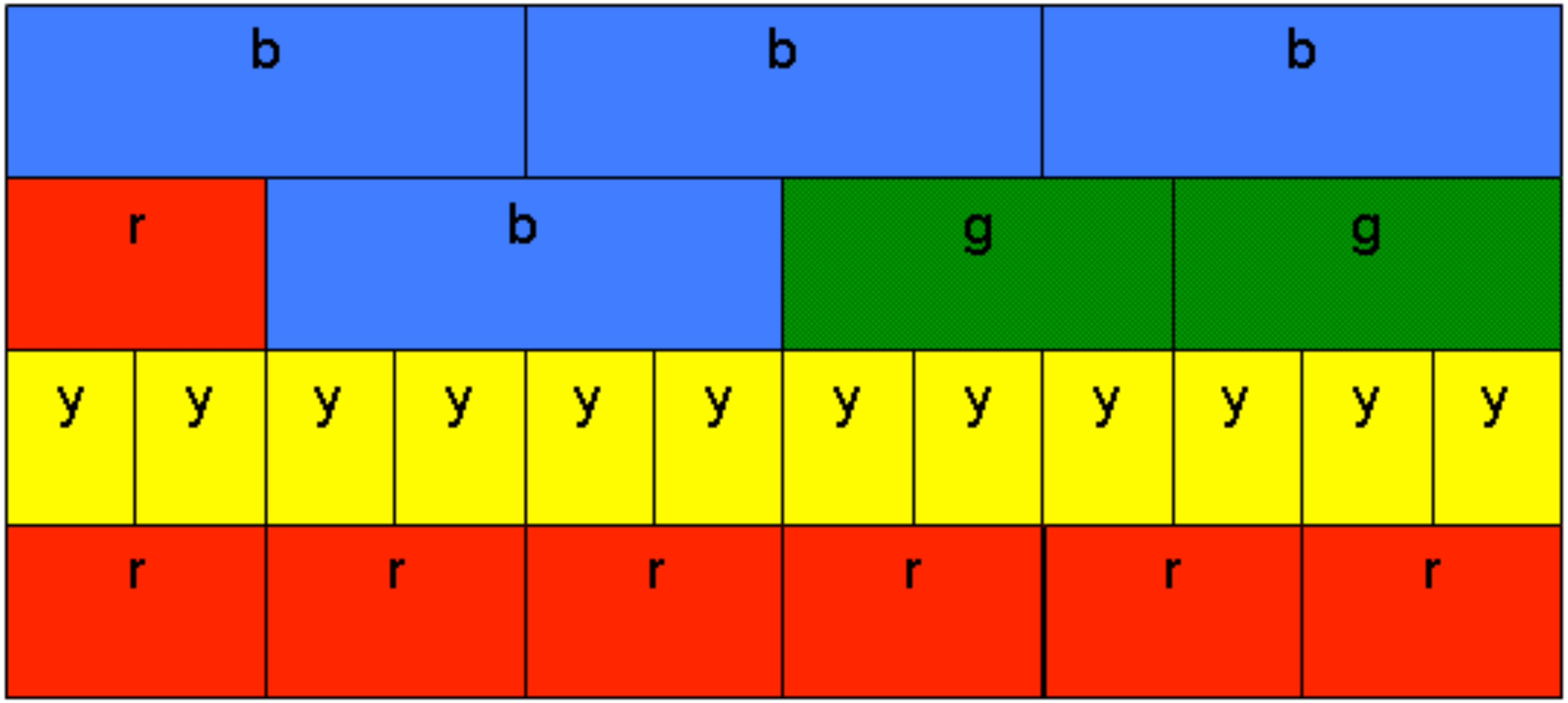
Forming your own Expressions

Now, once you have got to grips with Rules 1 - 3, you should be able to have a go at forming your very own algebraic expressions.

Have a look at the following diagram and see if you can figure out where I have got the expressions below from. I am sure you could make up some much better ones.



$b = 2r$ $\quad b = 4y$ $\quad 2g = 3r$ $\quad b + r = 6y$ $\quad 3r - 3y = g$

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Now sometimes you are told a story and asked to form an expression using the information you are given. No problem, so long as you understood Rules 1 - 3

Once upon a time, in the Land of Algebra

Mr Barton has been sent has been sent on a shopping trip by his girlfriend and he is trying to figure out how much money he needs to bring.

A glance down the list reveals he needs 5 pears, 2 tins of beans, and a box of chocolates.

What is the total cost of these items?

Well, until I get there I don't know how much each item will cost, so I'll need to make up some letters… hmm… how about **p** for the price of pears, **b** for the price of beans, and **g** for the price of the box of chocolates (because I know she likes Galaxy!). It does not matter at all which you choose!

So what's the total cost so far?…

Well, 5 pears and each one costs **p** → **5p**

2 tins of beans each costing **b** → **2b**

and a box of chocolates costing **g** → **g**

So, Total Cost = $5p + 2b + g$

Note: the terms are DIFFERENT so don't try and add them together!

Then I get a text saying we need two more tins of beans and one less pear. Now how much will it cost?

So, Total Cost = $5p + 2b + g + 2b - p = 4p + 4b + g$

Then she announces that her parents are coming around, so we need twice as much of everything! Cost?

So, Total Cost = $2 \times (4p + 4b + g) = 8p + 8b + 2g$

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Substitution

One other thing the Rules of Algebra allow you to do is to substitute numbers into expressions. For this, you need to remember our friends BODMAS and Negative Numbers!

If: $a = 2$ $b = 5$ $c = -3$ $d = -10$

Work out the values of the following expressions:

$$3ab$$

Well this means:

$$3 \times a \times b$$

Using our values:

$$3 \times 2 \times 5 = 30$$

$$2ac - acd$$

Okay, so we have to do our multiplications first

$$2ac = 2 \times 2 \times -3 = -12$$

$$acd = 2 \times -3 \times -10 = 60$$

So together we have: $-12 - 60 = -72$

$$6cb$$

Well this means:

$$6 \times c \times b$$

Using our values:

$$6 \times -3 \times 5 = -90$$

$$5ad^2$$

Okay, we must sort out our power first:

$$d^2 = -10 \times -10 = 100$$

Now we can multiply together

$$5 \times 2 \times 100 = 1000$$

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2. Single Brackets

Once Upon A Time...

I once heard my very first teaching mentor in Nottingham explaining a very nice way of thinking about brackets.

He said to think of the brackets as a canoe, and to think of the term outside them as a wave. Now, as you know, when you are in a canoe, there is no place to hide from the wave, and the person at the back gets just as wet as the person at the front and those in the middle.

Which brings us nicely onto the single most important rule of brackets...

Key Rule: you must multiply EVERYTHING inside the bracket by the term on the outside

And so long as you remember this, as well as your Rules of Algebra and how to deal with Negative Numbers, then this topic should hold no fear for you!

I am going to take you through 4 pretty easy examples to make sure your knowledge of negative numbers and the rules of algebra is up to scratch, and then it's time for a few stinkers!

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Example 1

$$3(2a+6)$$

Okay, so remember, the 3 is multiplying the 2a AND the +6.

Sometimes drawing on arrows helps you remember this, and a box is useful too...

$$3\,(2a\,\boxed{+\ \ 6})$$

And so we get...

$$3 \times 2a = 6a$$

$$3 \times 6 = 12$$

Now, we are close to our answer, but we are missing... a SIGN

You must remember your rules for multiplying with negative numbers

The 3 and the front is really +3, and the second term in the bracket is +6, and two positives multiplied together give a POSITIVE so...

$$3(2a+6) = 6a+12$$

Example 2

$$5(7d-4)$$

Okay, so remember, the 5 is multiplying the 7a AND the -4.

Let's get those arrows going again, and a box too to remind us that the 2nd term in the bracket is a -4

$$5\,(7d\,\boxed{-\ \ 4})$$

And so we get...

a positive × a negative

$$5 \times 7\mathrm{d} = 35a$$

$$5 \times -4 = -20$$

And now we have our answer, but notice again how important it was to get the sign correct.

If I had £1 for each time I have seen 35a + 20, or just 35a 20 for questions like this, Mr Barton would be loaded!

Anyway, the correct answer...

$$5(7d-4) = 35d-20$$

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Example 3

$$-4(t+2)$$

Okay, so remember, the -4 is multiplying the t AND the +2.

Arrows and boxes...

$$-4\ (t\ \boxed{+\ 2})$$

And so we get...

a negative
× a positive

$$-4 \times t = -4t$$
$$-4 \times 2 = -8$$

So long as you are good with negative numbers, you should have been able to get those signs correct!

$$-4(t+2) = -4t-8$$

Example 4

$$-10(2c-4)$$

Okay, so remember, the -10 is multiplying the 2c AND the -4.

Arrows and boxes...

$$-10\ (2c\ \boxed{-\ 4})$$

Be careful with your signs...

$$-10 \times 2c = -20c$$
$$-10 \times -4 = 40$$

a negative ×
a negative

The 2nd multiplication always catches people out. Remember, two negatives multiplied together give a POSITIVE!

$$-10(2c-4) = -20c+40$$

Time for the stinkers...

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Example 5

$$5a(2b-c)$$

Okay, so remember, the 5a is multiplying the 2b AND the -c.

Arrows and boxes...

$$5a\ (2b\ \boxed{-\ c})$$

You need Rules of Algebra and Negative Numbers for this...

$$5a \times 2b = 10ab$$

$$5a \times -c = -5ac$$

If you didn't follow any of that, make sure you go back and read over the 1. Rules of Algebra notes again!

Example 6

$$7ar(10st+2b-5)$$

Okay, so remember, the 7ar is multiplying the 10st AND the +2b AND the -5.

Arrows and boxes...

$$7ar\ (10st\ \boxed{+\ 2b}\ \boxed{-\ 5})$$

Be careful with your signs and letters...

$$7ar \times 10st = 70arst$$

$$7ar \times 2b = 14abr$$

$$7ar \times -5 = -35ar$$

Again, if you missed any of that, you know what to do...

$$5a(2b-c) = 10ab-5ac \qquad 7ar(10st+2b-5) = 70arst+14abr-35ar$$

Too easy for you?...

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Example 7

$$4r(2r-9t)$$

Okay, so remember, the 4r is multiplying the 2r AND the -9t.

Arrows and boxes…

$$4r\ (2r\ \boxed{-\ 9t})$$

You definitely need your Rules of Algebra for this…

$$4r \times 2r = 8rr = 8r^2$$

$$4r \times -9t = -36rt$$

The first one was the tricky bit there! Something, multiplied by itself, becomes squared!

$$4r(2r-9t) = 8r^2-36rt$$

Example 8

$$2ab(4a-3ab^2)$$

Okay, so remember, the 2ab is multiplying the 4a AND the $-3ab^2$

Arrows and boxes…

$$2ab\ (4a\ \boxed{-\ 3ab^2})$$

How well do you know your algebra?…

$$2ab \times 4a = 8aab = 8a^2b$$

$$2ab \times -3ab^2 = -6aabbb = -6a^2b^3$$

That's about as difficult as they get!

$$2ab(4a-3ab^2) = 8a^2b-6a^2b^3$$

And I think that'll do!

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3. Factorising

What on earth does Factorising mean?...

Very simply, factorising is the opposite of what we did in the previous section – 2. Brackets

Factorising just means: putting back into brackets

How to Factorise

1. Look for the highest common factors in each term (they could be letters or numbers)

2. Place these common factors outside the bracket

3. Write down what is now left inside the bracket – ask yourself: what do I need to multiply the term outside the bracket by to get my original term?

4. Check carefully that there are no more common factors in your bracket

5. Check your answer by expanding your brackets – it takes 2 seconds and it means you have definitely got the question correct!

Let's make sure we understand about Factors...

The key to successful factorising is understanding factors, and if it helps, why not just write down what each term means in full and then it's dead easy to spot the factors...

$$12a \longrightarrow 12 \times a$$

$$6y^2 \longrightarrow 6 \times y \times y$$

$$7pq^2 \longrightarrow 7 \times p \times q \times q$$

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Example 1

Factorise: $7a+21$

1. Okay, so we're on the hunt for common factors in both numbers and letters:

Numbers: 7 and 21 → Highest Factor = 7

Letters: there are no letters in the 2nd term, so we can't take any letters outside the bracket!

2. So we have…

$$7\,(\;?\;+\;?\;)$$

3. Now we have to figure out…

$$7 \times ? = 7a \longrightarrow a$$

$$7 \times ? = 21 \longrightarrow 3$$

Which gives us… $7(a + 3)$

4. Check there are no more common factors left inside the bracket…erm… nothing is common to both a and 3, so we're fine!

5. Expand the answer (on paper or in your head) to make sure you get the original question!

Example 2

Factorise: $10p+15pq$

1. Okay, so we're on the hunt for common factors in both numbers and letters:

Numbers: 10 and 15 → Highest Factor = 5

Letters: p and p q → Highest Factor = p

2. So we have…

$$5p\,(\;?\;+\;?\;)$$

3. Now we have to figure out…

$$5p \times ? = 10p \longrightarrow 2$$

$$5p \times ? = 15pq \longrightarrow 3q$$

Which gives us… $5p\,(2 + 3q)$

4. Check there are no more common factors left inside the bracket…erm… nothing is common to both 2 and 3q, so we're fine!

5. Expand the answer (on paper or in your head) to make sure you get the original question!

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Example 3

Factorise: $24c^2 + 16c$

1. Okay, so we're on the hunt for common factors in both numbers and letters:

Numbers: 24 and 16 → Highest Factor = 8

Letters: c^2 and c → Highest Factor = c

Remember: c^2 is just $c \times c$

2. So we have...

$$8c\ (\ ?\ +\ ?\ )$$

3. Now we have to figure out...

$$8c \times ? = 24c^2 \longrightarrow 3c$$

$$8c \times ? = 16c \longrightarrow 2$$

Which gives us... $8c(3c + 2)$

4. Check there are no more common factors

5. Expand the answer (on paper or in your head) to make sure you get the original question!

NOTE:

A very common mistake is not to take out the highest common factor.

For example, imagine we were doing Example 3, but for the numbers we thought the highest common factor was 2...

Numbers: 24 and 16 → Highest Factor = 2

Letters: c^2 and c → Highest Factor = c

We would get...

$$2c\ (\ ?\ +\ ?\ )$$

And then...

$$2c \times ? = 24c^2 \longrightarrow 12c$$

$$2c \times ? = 16c \longrightarrow 8$$

Which gives us... $2c(12c + 8)$

But, so long as we remember to always check there are no more common factors, we'll be fine, because a quick glance at this answers shows us that 12 and 8 have a common factor of 4!

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Example 4

Factorise: $18bc - 45b^2$

1. Okay, so we're on the hunt for common factors in both numbers and letters:

Numbers: 18 and 45 → Highest Factor = 9

Letters: b c and b^2 → Highest Factor = b

Remember: b^2 is just b × b and b c is just b × c

2. So we have...

$$9b\ (\ ?\ -\ ?\ )$$

3. Now we have to figure out...

$$9b \times ? = 18bc \longrightarrow 2c$$

$$9b \times ? = 45b^2 \longrightarrow 5b$$

Which gives us... $9b(2c - 5b)$

4. Check there are no more common factors

5. Expand the answer (on paper or in your head) to make sure you get the original question!

Example 5 - Nightmare!

Factorise: $18a^2b - 6ab + 30ab^2$

1. Okay, so we're on the hunt for common factors in both numbers and letters:

Numbers: 18 6 and 30 → Highest Factor = 6

Letters: a^2 b a b and a b^2 → Highest Factor = a b

Remember: a^2 b is just a × a × b and
a b^2 is just a × b × b

2. So we have...

$$6ab\ (\ ?\ -\ ?\ -\ ?\ )$$

3. Now we have to figure out...

$$6ab \times ? = 18a^2b \longrightarrow 3a$$

$$6ab \times ? = 6ab \longrightarrow 1$$

$$6ab \times ? = 30ab^2 \longrightarrow 5b$$

Which gives us... $6ab(3ac - 1 + 5b)$

4/5. Check for common factors and Expand the answer to make sure you are correct!

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4. Solving Linear Equations

What on earth does Solving Equations mean?

Let's look at each of these three words in turn…

Equations – these are just the same as expressions (what we have looked at in the last 3 sections, but with an equals sign (=) thrown in for good measure

Linear – this just means we don't have to worry about annoying powers… just yet!

Solving – this means we must find the value of the unknown which makes the equation balance

Now, there are a lot of different ways to solve equations, and if you are happy with the way that you have been taught, then stick to it, but this is the way I do them…

How Mr Barton Solves Equations

Golden Rule: Whatever you do to one side of the equation, you must do exactly the same to the other side to keep the equation in balance

Aim: To be left with your unknown letter on one side of the equals sign, and a number on the other side

Method:

By doing the same to both sides of the equation…

1. If they are not already, get all your unknown letters on one side of the equation (NOT on the bottom of fractions and avoiding negatives).
2. Begin unwrapping your unknown letter, by thinking about the order that things were done to the letter
3. Use inverse operations to do this until you are left with just your unknown letter on one side, and the answer on the other
4. Check your answer using substitution and you should never ever get one of these wrong!

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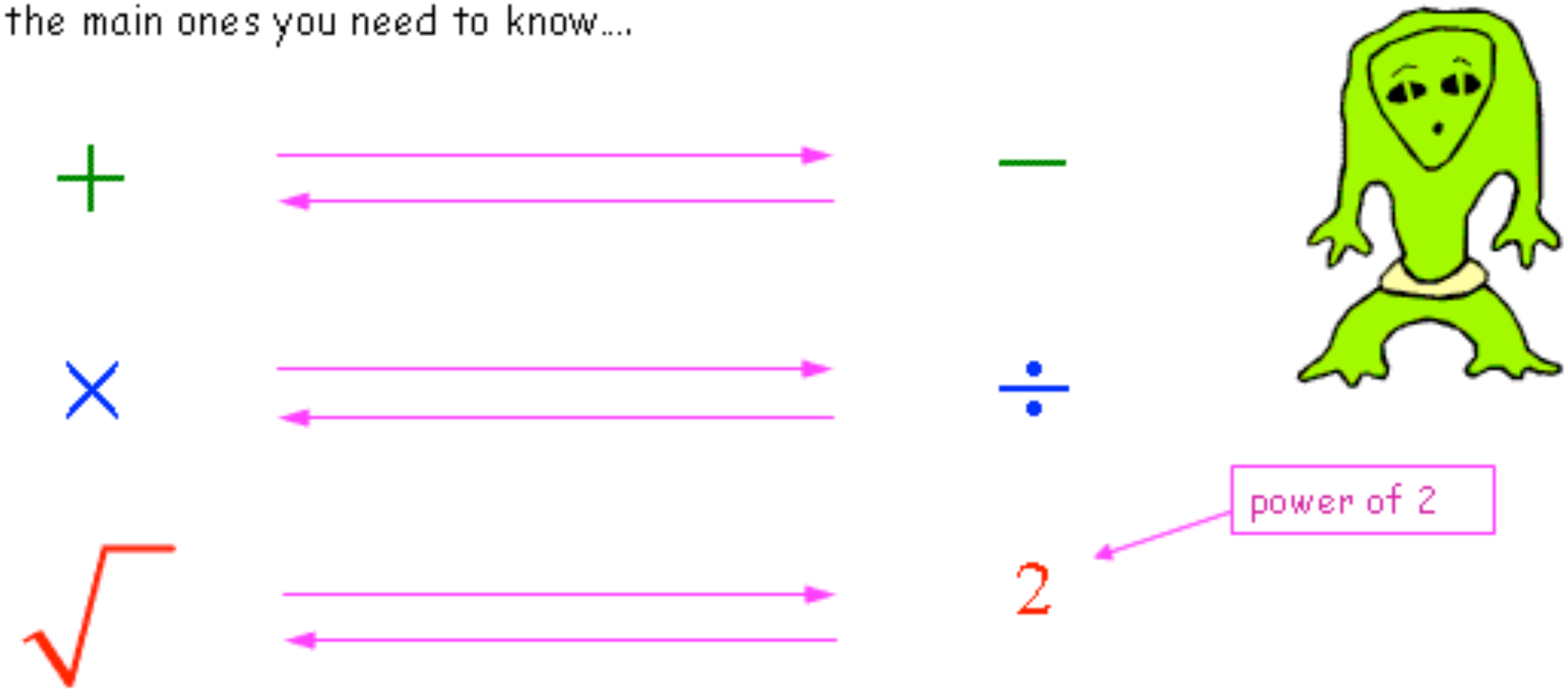
What are Inverse Operations?...

Inverse operations are the key to solving equations as they allow you to unwrap all the things surrounding your unknown letter and leave you with a simple answer.

Inverse operations are just operations which are the opposite of each other, and as such they cancel each other out.

Here are the main ones you need to know....



Now, the way I am going to set out these first few examples may seem very long and painful, but if you can do it this way for the simple ones, there is no reason why you can do the same for the nightmare stinker ones at the end...

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Example 1 $7p - 3 = 32$

1. Right, here we go… now our unknown letter (p) only appears on the left hand side of the equation, there is no negative sign in front of it, and it is not on the bottom of a fraction, so that's a good start!

$$7p - 3 = 32$$

2. Okay, what order were things done to p…

$$p \xrightarrow{\times 7} 7p \xrightarrow{-3} 7p - 3$$

3. And so now we can unwrap, starting with the last operation, and doing the inverse (opposite) to both sides:

Add three to both sides

Notice how the +3 cancels out the -3!

$+3$

$$7p - 3 + 3 = 32 + 3$$

$$\rightarrow \quad 7p = 35$$

Divide both sides by 7

Notice how dividing by 7 cancels out the 7 multiplying the p

$\div 7$

$$\frac{7p}{7} = 35 \div 7$$

$$\rightarrow \quad p = 5$$

4. We have our answer, but it's so easy to check if we are right, that we might as well do it.

Just substitute p = 5 into the questions, and hope the equation balances…

When p = 5…

$$7p - 3 = 7 \times 5 - 3 = 35 - 3 = 32!$$

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Example 2 $2(3r + 6) = 36$

1. Okay, so let's do our checks… our unknown letter (r) only appears on the left hand side of the equation, there is no negative sign in front of it, and it is not on the bottom of a fraction, so we are good to go… after we expand the brackets, of course…

$$2(3r + 6) = 36$$

$$\rightarrow \quad 6r + 12 = 36$$

Note: if this bit confused you, have another read of 1. Rules of Algebra

2. Okay, what order were things done to r…

$$r \xrightarrow{\times 6} 11r \xrightarrow{+12} 6r + 12$$

3. And so now we can unwrap, starting with the last operation, and doing the inverse (opposite) to both sides:

Subtract twelve from both sides

Notice how the -12 cancels out the +12!

$$-12 \quad 6r + 12 - 12 = 36 - 12$$

$$\rightarrow \quad 6r = 24$$

Divide both sides by 6

Notice how dividing by 6 cancels out the 6r!

$$\div 6 \quad \frac{6r}{6} = 24 \div 6$$

$$\rightarrow \quad r = 4$$

4. We have our answer, but it's so easy to check if we are right, that we might as well do it.

Just substitute r = 4 into the questions, and hope the equation balances…

When r = 4…

$$2(3r + 6) = 2(3 \times 4 + 6) = 2(12 + 6) = 2 \times 18 = 36!$$

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Example 3 $6+\frac{k}{5} = -1$

1. Okay, so our unknown letter (k) only appears on the left hand side of the equation, there is no negative sign in front of it, and it is not on the bottom of a fraction. Phew!

$$6+\frac{k}{5} = -1$$

2. Okay, what order were things done to k…

$$k \xrightarrow{\div 5} \frac{k}{5} \xrightarrow{+6} 6+\frac{k}{5}$$

Note: just because k is not written first, doesn't change the order in which things are done to k! Think: BODMAS!

3. And so now we can unwrap, starting with the last operation, and doing the inverse (opposite) to both sides:

Subtract six from both sides

-6

$$6+\frac{k}{5}-6 = -1-6$$

Again, look at the cancelling out!

$$\rightarrow \frac{k}{5} = -7$$

Multiply both sides by 5

$\times 5$

$$\frac{k}{5}\times 5 = -7\times 5$$

It all cancels out!

$$\rightarrow k = -35$$

We have our answer, but it's so easy to check if we are right, that we might as well do it.

Just substitute k = -35 into the questions, and hope the equation balances…

When k = -35…

$$6+\frac{k}{5} = 6+\frac{-35}{5} = 6+-7 = 6-7 = -1!$$

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Example 4 $24 - 3m = 6$

1. Okay, so let's do our checks… our unknown letter (m) only appears on the left hand side of the equation, it's not on the bottom of a fraction, but wait… it's got a negative sign in front of it!

This is going to make life difficult, but we can sort it out by using inverse operations to cancel out the -3m…

We just need to add 3m to both sides!

And now we have an equation just like all the others!

$$24 - 3m = 6$$

$+3m$

$$24 - 3m + 3m = 6 + 3m$$

$$\rightarrow \quad 24 = 6 + 3m$$

2. Okay, what order were things done to m…

$$m \xrightarrow{\times 3} 3m \xrightarrow{+6} 6 + 3m$$

3. And so now we can unwrap, starting with the last operation, and doing the inverse (opposite) to both sides:

Subtract six from both sides

The 6s on the right hand side will cancel!

-6

$$24 - 6 = 6 + 3m - 6$$

$$\rightarrow \quad 18 = 3m$$

Divide both sides by 3

$\div 3$

$$18 \div 3 = \frac{3m}{3}$$

$$\rightarrow \quad 6 = m \quad \text{or} \quad m = 6$$

Substitution to check our answer: When m = 6…

$$24 - 3m = 24 - 3 \times 6 = 24 - 18 = 6!$$

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Example 5 $7y + 3 = 10y - 6$

$$7y + 3 = 10y - 6$$

1. Okay, we have trouble right away! All of the unknowns (y) are NOT on the same side.

No problem, we just need a bit of inverse operations.

Top Tip: Collect your letters on the side which starts off with the most letters... so the right hand side!

So, we just need to subtract 7y from both sides!

And now we have an equation just like all the others!

$-7y$

$$7y + 3 - 7y = 10y - 6 - 7y$$

$$\rightarrow \quad 3 = 3y - 6$$

Note: if this bit confused you, have another read of 1. Rules of Algebra

2. Okay, what order were things done to y...

$$y \xrightarrow{\times 3} 3y \xrightarrow{-6} 3y - 6$$

3. And so now we can unwrap, starting with the last operation, and doing the inverse (opposite) to both sides:

Add six to both sides

The 6s on the right hand side will cancel!

$+6$

$$3 + 6 = 3y - 6 + 6$$

$$\rightarrow \quad 9 = 3y$$

Divide both sides by 3

$\div 3$

$$9 \div 3 = \frac{3y}{3}$$

$$\rightarrow \quad 3 = y \quad \text{or} \quad y = 3$$

Substitution to check our answer balances!: When y = 3...

Left hand side $7y + 3 = 7 \times 3 + 3 = 24$

Right hand side $10y - 6 = 10 \times 3 - 6 = 24$

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Example 6 $\frac{25}{g-1} = 5$

1. Problem! Our unknown letter (g) is on the bottom of a fraction!

$$\frac{25}{g-1} = 5$$

The only way we are going to get that g off the bottom of the fraction is to realise that the 25 is being divided by g - 1 and use inverse operations…

So, we just need to multiply both sides by (g - 1)

$\times(g-1)$

$$\frac{25}{g-1} \times (g-1) = 5 \times (g-1)$$

And expand the brackets on the right hand side

$$\rightarrow 25 = 5(g-1)$$

And now we have an equation just like all the others!

$$\rightarrow 25 = 5g - 5$$

2. Okay, what order were things done to g…

$$g \xrightarrow{\times 5} 5g \xrightarrow{-5} 5g-5$$

3. And so now we can unwrap, starting with the last operation, and doing the inverse (opposite) to both sides:

Add five to both sides

$+6$

$$25 + 5 = 5g - 5 + 5$$

The 5s on the right hand side will cancel!

$$\rightarrow 30 = 5g$$

Divide both sides by 5

$\div 5$

$$30 \div 5 = \frac{5g}{5}$$

$$\rightarrow 6 = g \quad \text{or} \quad g = 6$$

Substitution to check our answer is correct!: When g = 6…

$$\frac{25}{g-1} = \frac{25}{6-1} = \frac{25}{5} = 5$$

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5. Double Brackets

You knew it was coming...

Just when you have got your head around how to expand single brackets, your lovely maths teacher announces it's time to have a go at expanding double brackets.

But the good news is that it's no more difficult than single brackets, you don't need to learn any new skills, and you get loads more marks for doing it!

Skills you need for success...

If you know about these things, you will be fine:

- How to expand single brackets (see Algebra 2. Single Brackets)
- Rules of Algebra (see Algebra 1. Rules of Algebra)
- Rules of Negative Numbers (see Number 8. Negative Numbers)

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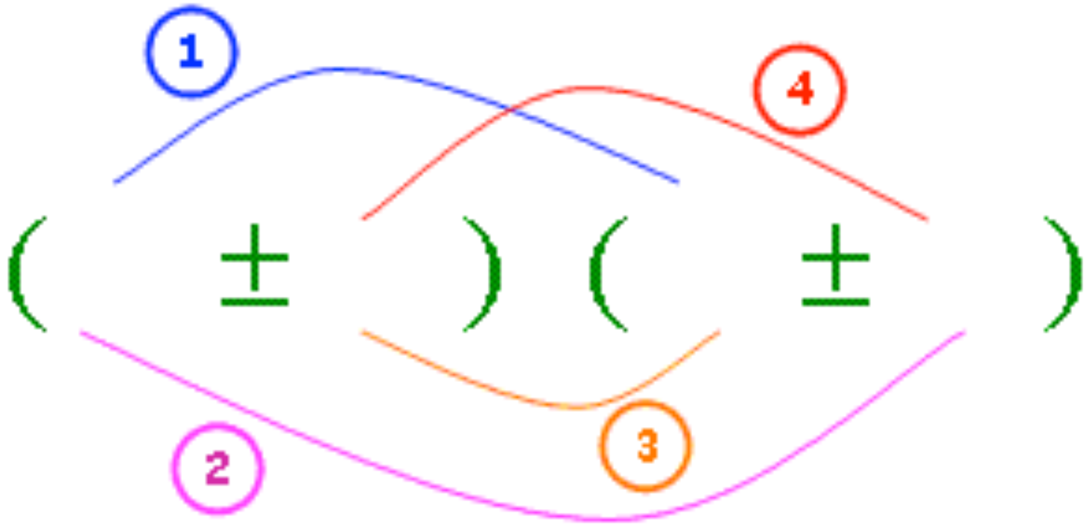
It's all about FOIL…

Now, like with most things in maths, there are a lot of different ways of expanding double brackets, and if you are happy with your way, then just stick to it, but here is how I do it.

FOIL basically tells me the order in which I need to multiply terms, because the most common mistake people make when expanding double brackets is to miss a few terms out!



Some people call this the smiley face method!

1.	First	Multiply together the first terms in each bracket – remembering to include the signs in front of them
2.	Outer	Multiply together the terms on the outside each bracket – remembering to include the signs in front of them
3.	Inner	Multiply together the terms on the inside each bracket – remembering to include the signs in front of them
4.	Last	Multiply together the last terms in each bracket – remembering to include the signs in front of them

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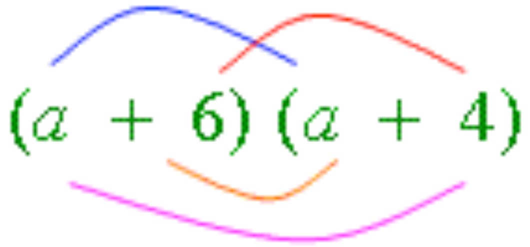
Example 1

$$(a + 6)(a + 4)$$

Until you get really comfortable, there is nothing wrong with drawing the smiley face on to remind you what to multiply!



$$(a + 6)(a + 4)$$

First $a \times a = a^2$

Outer $a \times 4 = 4a$

Inner $6 \times a = 6a$

Last $6 \times 4 = 24$

Now we write down our answers, in order, remembering if there is no sign in front of our term it's just a disguised plus!

$$a^2 \boxed{+ 4a + 6a} + 24$$

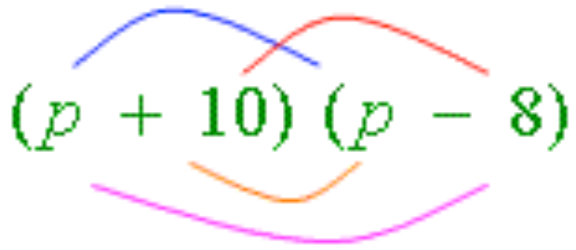
Notice that the middle terms simplify to give...

$$a^2 + 10a + 24$$

Example 2

$$(p + 10)(p - 8)$$

Time for the smiley face...



$$(p + 10)(p - 8)$$

Be really careful with the NEGATIVES...

First $p \times p = p^2$

Outer $p \times -8 = -8p$

Inner $10 \times p = 10p$

Last $10 \times -8 = -80$

Now we write down our answers, in order, making sure we get all the signs correct!

$$p^2 \boxed{- 8p + 10p} - 80$$

Notice that the middle terms simplify to give...

$$p^2 + 2p - 80$$

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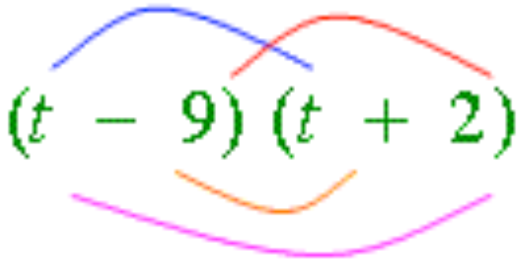
Example 3

$$(t - 9)(t + 2)$$

Let's draw our smiley face...



$$(t - 9)(t + 2)$$

Again, we must watch those NEGATIVES...

First $t \times t = t^2$

Outer $t \times 2 = 2t$

Inner $-9 \times t = -9t$

Last $-9 \times 2 = -18$

Once again, the signs are the key to success!

$$t^2 \boxed{+ 2t - 9t} - 18$$

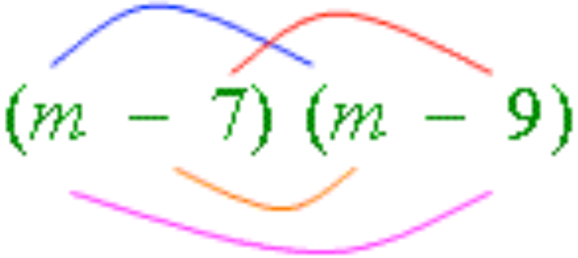
Carefully simplify the middle terms...

$$t^2 - 7t - 18$$

Example 4

$$(m - 7)(m - 9)$$

Time for another smiley face...



$$(m - 7)(m - 9)$$

Be so, so, so careful with the NEGATIVES...

First $m \times m = m^2$

Outer $m \times -9 = -9m$

Inner $-7 \times m = -7m$

Last $-7 \times -9 = 63$

Writing down our answers, we get...

$$m^2 \boxed{- 9m - 7m} + 63$$

You have to know your Rules of Negative Numbers inside out for this next bit...

$$m^2 - 16m + 63$$

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Let's take a moment to reflect...

Just before we look at a few more difficult ones (which, by the way, follow the exact same rules as these), I just want to draw your attention to the answers we got...

$$(a+6)(a+4) \longrightarrow a^2+10a+24$$

$$(p+10)(p-8) \longrightarrow p^2+2p-80$$

$$(t-9)(t+2) \longrightarrow t^2-7t-18$$

$$(m-7)(m-9) \longrightarrow m^2-16m+63$$

Now, look at the numbers in the questions and the numbers in the answers.

Can you see a quick way of getting from one to the other?...

Don't worry if you can't, but if you can then you are one step ahead, because that is the key to success at 6. More Factorising, which is coming up soon...

But for now, how about some tricky expanding double bracket questions?...

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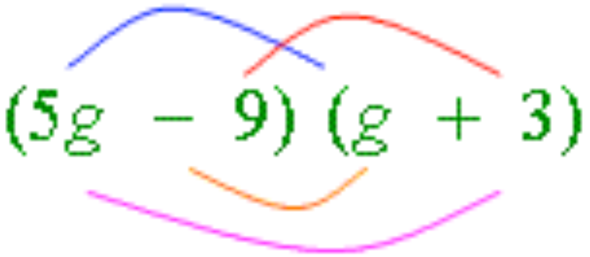
Example 5

$$(5g - 9)(g + 3)$$

Let's draw our smiley face...



$$(5g - 9)(g + 3)$$

Again, we must watch those NEGATIVES, and we must know our Rules of Algebra!

First $5g \times g = 5g^2$

Outer $5g \times 3 = 15g$

Inner $-9 \times g = -9g$

Last $-9 \times 3 = -27$

As always, the signs are the key to success!

$$5g^2 \boxed{+ 15g - 9g} - 27$$

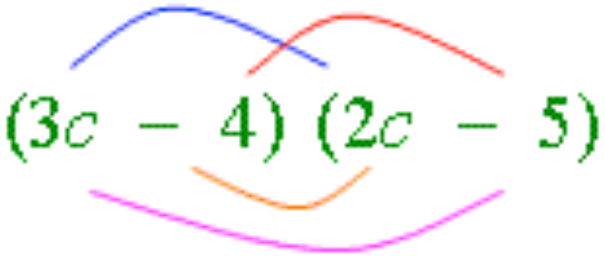
Carefully simplify the middle terms...

$$5g^2 + 6g - 27$$

Example 6

$$(3c - 4)(2c - 5)$$

Are you still feeling happy?...



$$(3c - 4)(2c - 5)$$

NEGATIVES and Rules of Algebra again...

First $3c \times 2c = 6c^2$

Outer $3c \times -5 = -15c$

Inner $-4 \times 2c = -8c$

Last $-4 \times -5 = 20$

Writing down our answers, we get...

$$6c^2 \boxed{- 15c - 8c} + 20$$

Carefully simplify the middle terms...

$$6c^2 - 23c + 20$$

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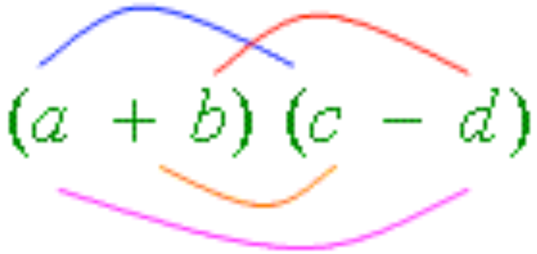
Example 7

$$(a + b)(c - d)$$

Let's draw our smiley face...



$$(a + b)(c - d)$$

Again, we must watch those NEGATIVES, and we must know our Rules of Algebra!

First $a \times c = ac$

Outer $a \times -d = -ad$

Inner $b \times c = bc$

Last $b \times -d = -bd$

As always, the signs are the key to success!

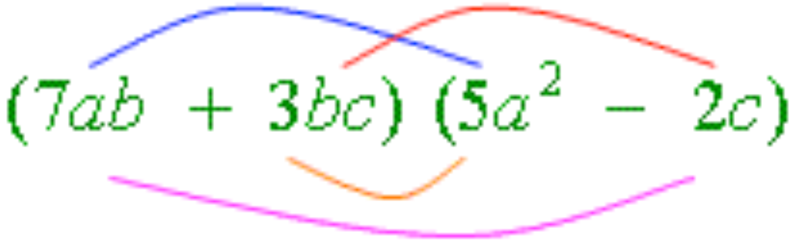
$$ac - ad + bc - bd$$

Can we simplify the middle two (or indeed, any) of the terms?... NO because there are NO LIKE TERMS!

Example 8 - because I am feeling nasty...

$$(7ab + 3bc)(5a^2 - 2c)$$

Are you still smiling now?...



$$(7ab + 3bc)(5a^2 - 2c)$$

Okay, you would be really unlucky to ever get one as hard as this, but there's no reason we can't do it

First $7ab \times 5a^2 = 35a^3b$

Outer $7ab \times -2c = -14abc$

Inner $3bc \times 5a^2 = 15a^2bc$

Last $3bc \times -2c = -6bc^2$

Phew! Writing down our answers, we get...

$$35a^3b - 14abc + 15a^2bc - 6bc^2$$

Can we simplify the middle two (or indeed, any) of the terms?... NO because there are NO LIKE TERMS!

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Last one, I promise...

How would you do this one?...

$$(a - 7)^2$$

If you said: "well, it's dead easy, isn't it, the answer is just...

$$a^2 - 49$$

Then please never say that again... because it's wrong!

Remember: squaring something means multiplying it by itself.
So, this question could actually be written as...

$$(a - 7)(a - 7)$$

Which means we can go back to our friend FOIL, and everyone is happy!
Incidentally, if you want to check you can still do these, the final simplified answer is...

$$a^2 - 14a + 49$$

Can you see how we could have reached that answer a quicker way?...

TO BE CONTINUED on a maths website near you...

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6. More Factorising – Quadratics

Again, you knew it was coming…

Just like we had to expand double brackets, it should come as no surprise that we have to factorise expressions back into double brackets as well!

There is a bit of a trick to this, and to discover it, let's look back at our answers from 5. Expanding Brackets…

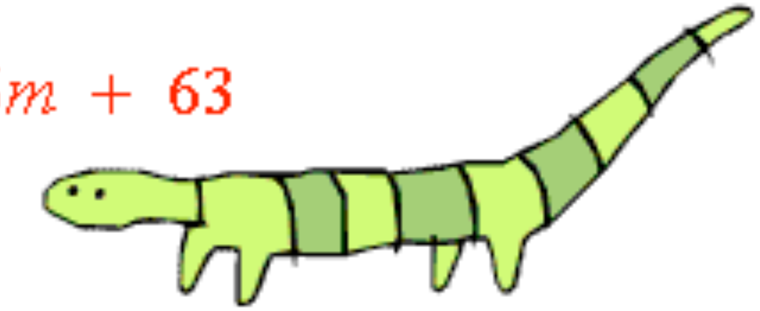
$$(a + 6)(a + 4) \longrightarrow a^2 + 10a + 24$$

$$(p + 10)(p - 8) \longrightarrow p^2 + 2p - 80$$

$$(t - 9)(t + 2) \longrightarrow t^2 - 7t - 18$$

$$(m - 7)(m - 9) \longrightarrow m^2 - 16m + 63$$

Note: These answers are called Quadratic Expressions because they have a squared term in them



Focus your attention on the numbers…

if you can see how to get from the numbers in the questions to the numbers in the answers…

then you should be able to see how to get from the answer back to the question…

and if you can do that, then you can already factorise quadratic expressions!

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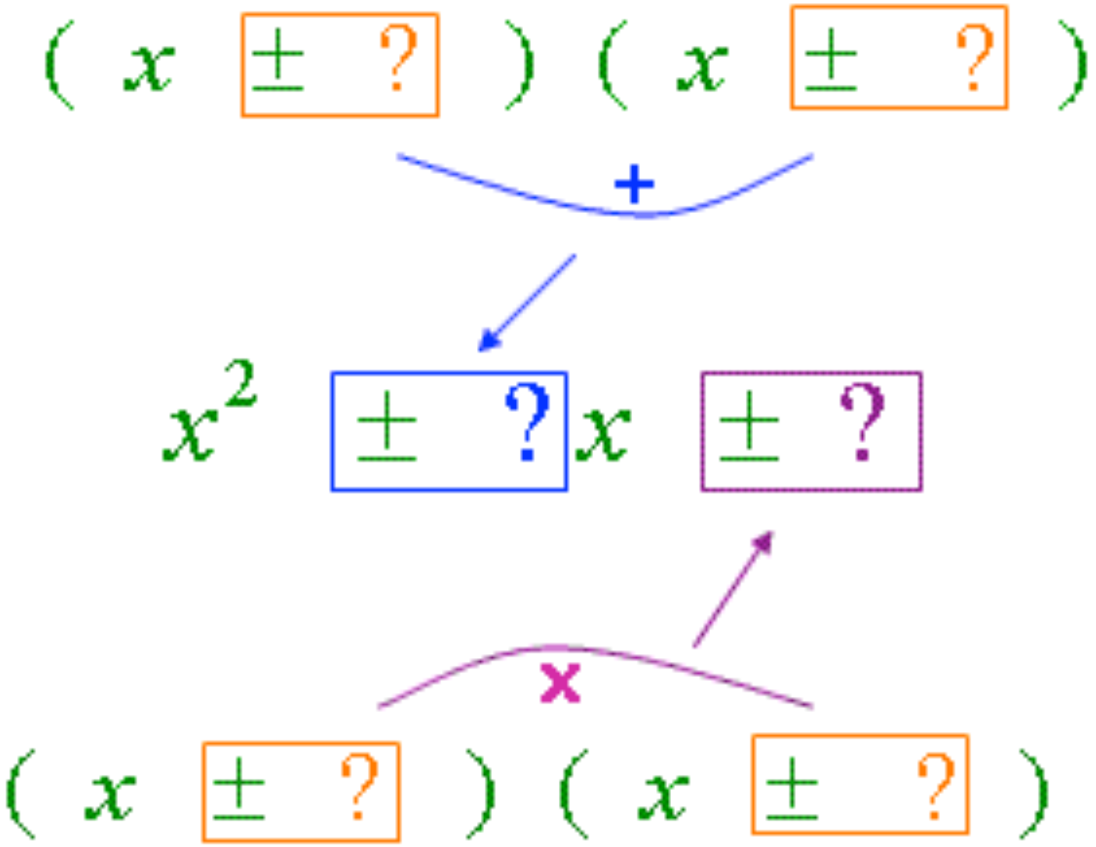
How to Factorise Quadratic Expressions

Factorising quadratics means you want to get from:

$$x^2 \boxed{\pm\ ?}x \boxed{\pm\ ?} \quad \text{to} \quad (\ x \boxed{\pm\ ?}\ )(\ x \boxed{\pm\ ?}\ )$$

To be able to do this you need to be able to solve a little puzzle

If you look back at the examples, you will see that ...



$$(\ x \boxed{\pm\ ?}\ )(\ x \boxed{\pm\ ?}\ )$$

$$x^2 \boxed{\pm\ ?}x \boxed{\pm\ ?}$$

$$(\ x \boxed{\pm\ ?}\ )(\ x \boxed{\pm\ ?}\ )$$

In other words, the two numbers in the bracket (including their sign) must...

ADD TOGETHER to give you the number (and sign) in front of the x...

And...

MULTIPLY TOGETHER to give you the number (and sign) at the end

So... if you can discover what two numbers solve that little puzzle, then you can factorise quadratics... and practice makes perfect!

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Example 1

$$x^2 + 11x + 24$$

Okay, here the question we must ask ourselves…

$$x^2 \boxed{+ \ 11}x \boxed{+ \ 24}$$

Which two numbers multiply together to give 24 and add together to give 11?

Now, if you find it helps, you can write down all the pairs of numbers which multiply together to give 24, and see which one also adds up to 11…

1×24	$1 + 24 = 25$	✗
2×12	$2 + 12 = 14$	✗
3×8	$3 + 8 = 11$	✓

Once we have our pair, we just write them in the brackets, remembering that no sign is just a disguised plus!

$(x + 3)(x + 8)$ or $(x + 8)(x + 3)$

Why not expand the brackets to make doubly sure you are correct!

Example 2

$$p^2 + 2p - 15$$

Okay, here the question we must ask ourselves…

$$p^2 \boxed{+ \ 2}p \boxed{- \ 15}$$

Which two numbers multiply together to give -15 and add together to give 2?

Again, nothing wrong with writing down pairs that multiply together to give -15, but be careful of your negatives!

1×-15	$1 + -15 = -14$	✗
-1×15	$-1 + 15 = 14$	✗
3×-5	$3 + -5 = -2$	✗
-3×5	$-3 + 5 = 2$	✓

Once we have our pair, we just write the numbers in the brackets, making sure we get our signs in the correct place!

$(p - 3)(p + 5)$ or $(p + 5)(p - 3)$

Again, expanding the brackets is a good check!

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Example 3

$$k^2 - 13k - 14$$

Okay, here the question we must ask ourselves…

$$k^2 \boxed{-\ 13}k \boxed{-\ 14}$$

Which two numbers multiply together to give -14 and add together to give -13?

Unless you can do it in your head, just write down pairs of numbers that multiply together to give -14:

-1×14 $\quad$ $-1 + 14 = 13$ ✗

1×-14 $\quad$ $1 + -14 = -13$ √

Once we have our pair, we just write the numbers in the brackets, making sure we get our signs in the correct place!

$$(k + 1)(k - 14) \text{ or } (k - 14)(k + 1)$$

If you expand the brackets you will definitely know that you are correct!

Example 4

$$v^2 - 9v + 18$$

Okay, here the question we must ask ourselves…

$$v^2 \boxed{-\ 9}v \boxed{+\ 18}$$

Which two numbers multiply together to give 18 and add together to give -9?

Now, switch on your brain here… we CAN'T be talking two positive numbers, as how will they add up to give -9?… so we need two negatives!

-1×-18 $\quad$ $-1 + -18 = -19$ ✗

-2×-9 $\quad$ $-2 + -9 = -11$ ✗

-3×-6 $\quad$ $-3 + -6 = -9$ √

People tend to mess these up, but we won't, because we know that two negatives multiplied together gives a positive!

$$(v - 3)(v - 6) \text{ or } (v - 6)(v - 3)$$

Again, expanding the brackets is a good check!

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What abut this funny looking one?...

$$x^2 - 16$$

Okay, looks a bit strange, but let's ask ourselves the same question as we always do...

Which two numbers multiply together to give -16 and add together to give... erm...well... erm... **0**?

Remember, it is the number in front of the x which tells us what the numbers must add together to make, but we don't have any x's, so the sum of our two numbers must be... **0!**

Think of the expression like this is it helps... $x^2 + 0x - 16$

So, isn't it true that for two numbers to add together to give zero, they must be the same number, but of opposite sign, so they cancel each other out!

So, which two numbers do we need?... 4 and -4! Expand it to check!

$$x^2 - 16 \longrightarrow (x + 4)(x - 4)$$

Expressions like this are called "the difference of two squares", and are always factorised in a similar way. Look at these three examples and see if you can see how I got the answers...

$$a^2 - 25 \longrightarrow (a + 5)(a - 5)$$

$$p^2 - 100 \longrightarrow (p + 10)(p - 10)$$

$$4t^2 - 49 \longrightarrow (2t + 7)(2t - 7)$$

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When things get a little tricky...

Okay, whilst it was not so tricky to spot how to factorise those types of quadratics, what about when there is a number in front of the squared term?...

Let's look back at two examples we did in 5. Expanding Double Brackets, to see if we can spot where the numbers come from...

$$(5g - 9)(g + 3) \longrightarrow 5g^2 + 6g - 27$$

$$(3c - 4)(2c - 5) \longrightarrow 6c^2 - 23c + 20$$

Any ideas?... It's not easy to spot, is it?

I think the best thing I can do is to try and take you through two examples as carefully as I can...

Are you ready?....

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Example 1 $2x^2 - x - 3$

Okay, let's start by thinking what the first terms in the two brackets must be?...

Do you agree that they would have to be **2x** and **x**, otherwise we would not get out **$2x^2$**!

How about the numbers at the end of each bracket?...

Well, they would have to be a pair of numbers which multiply together to give -3!

I set out this information in a table like this:

The first terms in the brackets

2x	-1	3	1	-3
x	3	-1	-3	1

All the pairs of numbers which multiply to give -3

Next I multiply DIAGONALLY, and I am looking for a pair of numbers which will add up to the amount of x's I need... which from the question is **-1x**!

2x	-1	3	-3	1
x	3	-1	1	-3

$(2x \times 3) + (x \times -1) = 6x + -1x = 5x$ ✗

$(2x \times -1) + (x \times 3) = -2x + 3x = 1x$ ✗

$(2x \times 1) + (x \times -3) = 2x + -3x = -1x$ √

The pair I want is the 3rd column of numbers along

Now, to get my answer, I just put the two terms from the top row in the first bracket, and the two terms from the second row in my second bracket...

$$(2x - 3)(x + 1)$$

You have done so much work here, that it is definitely worth checking you are correct by expanding the brackets!

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Example 2 $8x^2 - 2x - 15$

Okay, again let's start by thinking what the first terms in the two brackets must be?...

Problem: They could be either: **8x** and **x**, or the could be **4x** and **2x**. Both, when multiplied together, would give us the **$8x^2$** that we need!

So this time we need two tables!

The first terms in the brackets

All the pairs of numbers which multiply to give -15

8x	-1	15	1	-15	-3	5	3	-5
x	15	-1	-15	1	5	-3	-5	3

4x	-1	15	1	-15	-3	5	3	-5
2x	15	-1	-15	1	5	-3	-5	3

And once again I must just keep I multiplying DIAGONALLY (in my head if I can), looking for a pair of numbers which will add up to the amount of x's I need... which from the question is **-2x**!

After a long search, I reckon the pair with the rings around them is what I need!

$$(4x \times -3) + (2x \times 5) = -12x + 10x = -2x$$

As before, to get my answer I just put the two terms from the top row in the first bracket, and the two terms from the second row in my second bracket...

$$(4x + 5)(2x - 3)$$

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7. Solving Quadratic Equations

The three ways to solve quadratic equations…

Again, it will depend on your age and maths set as to how many of these you need to know, but here are the three ways which we can solve quadratic equations:

1. Factorising
2. Using the Quadratic Formula
3. Completing the Square

Which ever way you choose (or are told to do!), you must remember the Golden Rule:

The Golden Rule for Solving Quadratic Equations: You should always get TWO answers…
Note: in actual fact one (or even both) answer may not exist, but you don't need to worry about that until A Level!

Why on earth do I get two answers?…

This is all to do with the fact that quadratics contain squares, and what happens when we square negative numbers…

Imagine you were trying to think solve this equation: $x^2 = 25$

Well, $x = 5$ is definitely a solution that works, but there is another…erm… erm…
What about $x = -5$!… Because when you square a negative number, you get a positive answer!
And that's why we get two solutions when quadratics are involved!

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1. Solving by Factorising

This is by far the easiest and quickest way to solve a quadratic equation, and if you are not told otherwise, then always spend a minute or so seeing if the equation will factorise.

Note: For the rest of this section, I am going to assume you are comfortable with what was covered in Algebra 6. More Factorising. Please go back and have a quick read if not.

Method

1. Re-arrange the equation to make it equal to zero

2. Factorise the quadratic equation

3. Think what value of the unknown letter would make each of your brackets equal to zero

4. These two numbers are your answers!

Why on earth does that work?

Imagine, after following steps **1.** and **2.**, you find yourself looking at this...

$$(x - 4)(x + 3) = 0$$

Think about what we have got here... we have two things (x - 4) and (x + 3) that when multiplied together (disguised multiplication sign between the brackets) equal zero

Well... if two things multiplied together equal zero, then at least one of them must be zero!

So... you ask yourself: "what value of x makes the first bracket equal to zero?"... **4**!

And... "what value of x makes the second bracket equal to zero?"... **- 3**!

So we have our answers:

$$x = 4 \quad \text{or} \quad x = -3$$

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Example 1

$$x^2 - 3x - 28 = 0$$

Okay, let's go through each stage of the method:

1. The equation is already equal to zero, so that is a bonus!

2. Let's factorise the left hand side, like the good old days...

$$x^2 - 3x - 28 \rightarrow (x - 7)(x + 4)$$

And so, in terms of our equation, we have:

$$(x - 7)(x + 4) = 0$$

3. Right, we need to pick some values of x to make each of the brackets equal to zero:

$$\begin{tikzcd} (x - 7) \arrow[d] & (x + 4) \arrow[d] & = 0 \\ x = 7 & x = -4 & \end{tikzcd}$$

4. So, we have our answers...

$x = 7$ or $x = -4$

Example 2

$$2x^2 + 5x = 3$$

1. Problem: the equation is NOT equal to zero... but if we subtract 3 from both sides, we're good to go!

$$2x^2 + 5x - 3 = 0$$

2. This is one of the tricky factorisations...

$$2x^2 + 5x - 3 \rightarrow (2x - 1)(x + 3)$$

And so, in terms of our equation, we have:

$$(2x - 1)(x + 3) = 0$$

3. Right, we need to pick some values of x to make each of the brackets equal to zero:

$$\begin{tikzcd} (2x - 1) \arrow[d] & (x + 3) \arrow[d] & = 0 \\ x = 1/2 & x = -3 & \end{tikzcd}$$

4. So, we have our answers...

$x = 1/2$ or $x = -3$

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2. Solving by using the Quadratic Formula

The good news is that the quadratic formula can solve every single quadratic equations
The bad news is that it looks complicated and it's fiddly to use!

The Quadratic Formula:

+ or − this is where the 2 answers come from

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: To be able to use this formula, you must be very good at using your calculator. Practice to make sure you can get the answers I get below, and if not then ask you teacher!

What do the letters stand for?...

The letters are just the coefficients (the numbers in front of) the unknowns in your equation:
Remember: as always, you must include the signs of the numbers as well!

$$ax^2 + bx + c = 0$$

Example

$$5x^2 - 8x + 12 = 0 \longrightarrow a = 5 \qquad b = -8 \qquad c = 12$$

Never Ever Forget: Before you start sticking numbers into the formula, you must make sure that you rearrange your equation to make it equal to zero!

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Example 1

$$x^2 - 4x + 2 = 0$$

What a nice looking equation. I bet it factorises… erm… erm… no it doesn't!

So we'll have to use the formula.

It's already equal to zero, so we just need to figure out what our a, b and c are:

$$ax^2 + bx + c = 0$$

$$x^2 - 4x + 2 = 0$$

$$a = 1 \qquad b = -4 \qquad c = 2$$

Note: a = 1, and not 0! Remember, the 1 is hidden!

Stick the numbers in our formula…

$$x = \frac{--4 \pm \sqrt{-4^2 - 4\times1\times2}}{2\times1}$$

And if you are careful with your calculator, you should get…

$x = 3.41$ or $x = 0.59$ (2dp)

Pressing the buttons on the calculator

You'll be amazed how many people throw away easy marks because they can't use their calculator properly!

Here is one order of buttons you could press to get you the correct answer!

(-) ((-) 4) + √ (

Top Tip: always put any negative numbers in brackets or calculators tend to do daft things!

Change this to minus to work out the 2nd answer!

((-) 4) x² - 4 X

1 X 6) = 6.828427125

This gives you the value of the top of the fraction

÷ (2 X 1) =

3.414213562

Changing the sign to minus gives the other answer of…

0.585786437

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Example 2

$$5x^2 = 10 - 3x$$

It's not going to factorise, and it's not equal to zero!

So before we use the formula we must… add 3x and subtract 10 from both sides to give us:

$$5x^2 + 3x - 10 = 0$$

$$ax^2 + bx + c = 0$$
$$5x^2 + 3x - 10 = 0$$

$$a = 5 \qquad b = 3 \qquad c = -10$$

Stick the numbers in our formula…

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 5 \times -10}}{2 \times 5}$$

And if you are careful with your calculator, you should get…

$x = 1.15$ or $x = -1.75$ (2dp)

Pressing the buttons on the calculator

Okay, this time we'll work out the answer that uses the minus on top of the fraction instead of the plus…

((-) 3) - √ (

Top Tip: always put any negative numbers in brackets or calculators tend to do daft things!

Change this to plus to work out the 2nd answer!

3 x^2 - 4 X 5 X

((-) 1 0)) = -17.45683229

This gives you the value of the top of the fraction

÷ (2 X 5) =

-1.745683229

Changing the sign to plus gives the other answer of…

1.145683229

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3. Solving by Completing the Square

How would you factorise this?...

$$x^2 + 10x$$

It doesn't look like it can be done, but what about if I write it like this...

$$(x + 5)^2$$

Now that is certainly factorised as it is in brackets, but is it the correct answer?...
Let's expand the brackets using FOIL to find out...

$$(x + 5)^2 = (x+5)\,(x+5) = x^2+5x+5x+25 = x^2 + 10x + 25$$

It's close! In fact, our factorised version is just 25 too big! So, we can say...

$$x^2 + 10x = (x + 5)^2 - 25$$

And that is completing the square... the square is the **$(x + 5)^2$**, and the **- 25** completes it!

Method for Completing the Square

1. If the number in front of the x^2 is NOT 1, then take out a factor to make it so

2. Complete the Square using this fancy looking formula:

$$x^2 + bx = \left(x + {}^{b}\!/_{2}\right)^2 - \left({}^{b}\!/_{2}\right)^2$$

Note: b, is just the number (with sign!) in front of the x, like 10 in the example above!

3. If you need to solve the equation, use SURDS ... Crucial: When you square root, you must take both the positive and the negative to make sure you get TWO answers!

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Example 1 Complete the square and solve:

$$x^2 - 4x = 21$$

1. The number in front of x^2 is 1, so we're fine!

2. Let's use the formula on the left hand side:

$$x^2 + bx = \left(x + {}^{b}\!/_{2}\right)^2 - \left({}^{b}\!/_{2}\right)^2$$

$$x^2 - 4x = \left(x - {}^{4}\!/_{2}\right)^2 - \left({}^{-4}\!/_{2}\right)^2$$

$$\longrightarrow \quad (x - 2)^2 - 4$$

So now we have: $(x - 2)^2 - 4 = 21$

3. Well, so long as you are good at solving equations, you'll be fine from here...

$\boxed{+4}$ $\quad (x - 2)^2 = 25$

$\boxed{\surd}$ $\quad x - 2 = \pm\sqrt{25} = \pm 5$

$\boxed{+2}$ $\quad x = \pm 5 + 2$

So... $x = 5 + 2 = 7$ **Or**

$x = -5 + 2 = -3$

Example 2 Complete the square and solve:

$$4x^2 - 8x = 21$$

1. The number in front of x^2 is 4, so we must take out a factor of 4 to sort things out!

$$4\,(x^2 - 2x) = 21$$

2. Use the formula on the terms in the brackets:

$$x^2 - 2x = \left(x - {}^{2}\!/_{2}\right)^2 - \left({}^{-2}\!/_{2}\right)^2$$

$$\longrightarrow \quad (x - 1)^2 - 1$$

So now we have: $4\left[(x - 1)^2 - 1\right] = 21$

Expanding: $4\,(x - 1)^2 - 4 = 21$

3. Time to solve...

$\boxed{+4}$ $\quad 4(x - 1)^2 = 25$

$\boxed{\div 4}$ $\quad (x - 1)^2 = 6.25$

$\boxed{\surd}$ $\quad x - 1 = \pm\sqrt{6.25} = \pm 2.5$

$\boxed{+1}$ $\quad x = \pm 2.5 + 1$

$x = 2.5 + 1 = 3.5$ **Or** $x = -2.5 + 1 = -1.5$

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8. Simultaneous Equations

What are Simultaneous Equations?

Simultaneous Equations are two equations, each containing two unknown letters, and you have to use both equations, in a clever way, to find the value of your unknown letters!

Key Point: The values you find for your unknown letters must make BOTH equations balance – and once again this is another Algebra topic where you can check your answers and guarantee that you have got it right! I told you Algebra wasn't so bad...

Skills you need to have mastered before we start...

In this section I am going to assume that you are an world expert on the following things:

- How to solve equations (see Algebra 4. Solving Equations)
- Rules of Algebra (see Algebra 1. Rules of Algebra)
- Rules of Negative Numbers (see Number 8. Negative Numbers)

If this is not the case, go back now and have a quick read through!

Please Note: The graphical method for solving simultaneous equations is discussed in Graphs 1. Straight Line Graphs

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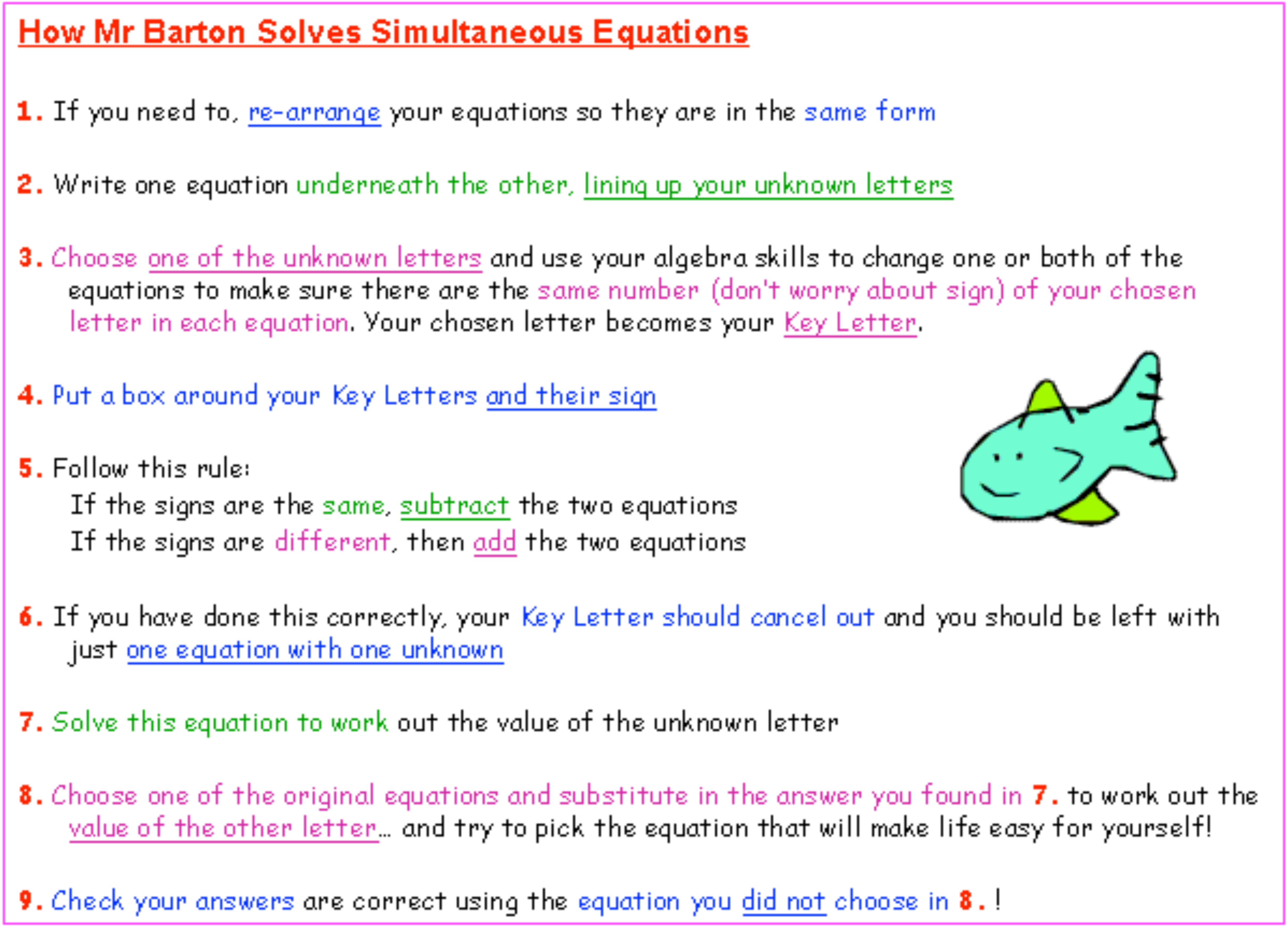

How Mr Barton Solves Simultaneous Equations

1. If you need to, re-arrange your equations so they are in the same form

2. Write one equation underneath the other, lining up your unknown letters

3. Choose one of the unknown letters and use your algebra skills to change one or both of the equations to make sure there are the same number (don't worry about sign) of your chosen letter in each equation. Your chosen letter becomes your Key Letter.

4. Put a box around your Key Letters and their sign

5. Follow this rule:
   If the signs are the same, subtract the two equations
   If the signs are different, then add the two equations

6. If you have done this correctly, your Key Letter should cancel out and you should be left with just one equation with one unknown

7. Solve this equation to work out the value of the unknown letter

8. Choose one of the original equations and substitute in the answer you found in **7.** to work out the value of the other letter... and try to pick the equation that will make life easy for yourself!

9. Check your answers are correct using the equation you did not choose in **8.** !

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Example 1 $3x + y = 19$ $x + y = 9$

1. Good news! Our equations are in the same form: some x's and some y's, equal a number!

2. Let's write the second equation underneath the first...

3. Okay, so we need to pick either the x's or the y's to be our Key Letter. Well... notice how there are already the same number of y's in both equations (there is a disguised 1 in front of both), so let's pick the y's to make life easier for ourselves!

4. Put a box around our Key Letters, and their signs:

5. The signs of our Key Letters are the same (both +) so we must Subtract equation ② from equation ①.

6. Our Key Letters have cancelled out, leaving us with a nice looking equation: $2x = 10$

7. Solve it:

8. Use this value in one of the original equations (I'll chose ①) to find the value of the other unknown letter:

9. And now we have our two answers: $x = 5$ $y = 4$

But we may as well check them using equation ②

② $x + y = 9$ $\boxed{\begin{matrix} x = 5 \\ y = 4 \end{matrix}} \rightarrow 5 + 4 = 9$

① $3x + y = 19$

② $x + y = 9$

$$\begin{array}{rrcl} \textcircled{1} & 3x \boxed{+\ y} & = & 19 \\ - \ \textcircled{2} & x \boxed{+\ y} & = & 9 \\ \hline & 2x & = & 10 \end{array}$$

$\boxed{\div 2} \rightarrow x = 5$

① $3x + y = 19$

$\boxed{x = 5} \rightarrow 15 + y = 19$

$\boxed{-15} \rightarrow y = 4$

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Example 2 $3x - 2y = 3 \qquad 2x + 2y = 12$

1. Good news! Our equations are in the same form: some x's and some y's, equal a number!

2. Let's write the second equation underneath the first...

3. Okay, so we need to pick either the x's or the y's to be our Key Letter. Well... notice how there are already the same number of y's in both equations (there 2 – don't worry about the sign!), so let's pick the y's to make life easier for ourselves!

4. Put a box around our Key Letters, and their signs:

5. The signs of our Key Letters are different (- and +) so we must Add equation ② to equation ①

6. Our Key Letters have cancelled out, leaving us with a nice looking equation: $5x = 15$

7. Solve it:

8. Use this value in one of the original equations (I'll chose ②) to find the value of the other unknown letter:

9. And now we have our two answers: $x = 3 \quad y = 3$

But we may as well check them using equation ①

① $3x - 2y = 3$ $\boxed{\begin{matrix} x = 3 \\ y = 3 \end{matrix}} \rightarrow 9 - 6 = 3$

① $3x - 2y = 3$

② $2x + 2y = 12$

$$\begin{array}{rl} ① & 3x \boxed{-\ 2y} = 3 \\ + \ ② & 2x \boxed{+\ 2y} = 12 \\ \hline & 5x \qquad\quad = 15 \end{array}$$

$\boxed{\div 5} \rightarrow x = 3$

② $2x + 2y = 12$

$\boxed{x = 3} \rightarrow 6 + 2y = 12$

$\boxed{-6} \rightarrow 2y = 6$

$\boxed{\div 2} \rightarrow y = 3$

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Example 3 $2x + 3y = 7 \qquad 3x + 5y = 18$

1. Good news! Our equations are in the same form: some x's and some y's, equal a number!

2. Let's write the second equation underneath the first…

3. Okay, bad news. We don't have the same number of either unknown. No problem, though! Why not make the number of x's the same by… multiplying ① by **3** and… multiplying ② by **2**
Note: We could have made the y's the same if we had liked!

4. Put a box around our Key Letters, and their signs:

5. The signs of our Key Letters are the same (disguised +) so we must Subtract equation ② from equation ①.

6. Our Key Letters have cancelled out, leaving us with a nice looking equation: $-y = -15$

7. Solve it (people seem to mess these ones up…)

8. Use this value in one of the original equations (I'll chose ②) to find the value of the other unknown letter:

9. And now we have our two answers: $x = -19 \quad y = 15$

But we may as well check them using equation ①

① $2x + 3y = 7$ $\boxed{\begin{array}{l} x = -19 \\ y = 15 \end{array}} \rightarrow -38 + 45 = 7$

① $2x + 3y = 7$

② $3x + 5y = 18$

① x 3 → $6x + 9y = 21$

② x 2 → $6x + 10y = 36$

$$\begin{array}{rl} ① & \boxed{6x} + 9y = 21 \\ - \;② & \boxed{6x} + 10y = 36 \\ \hline & -y = -15 \end{array}$$

$\boxed{\div\ -1} \rightarrow y = 15$

② $3x + 5y = 18$

$\boxed{y = -15} \rightarrow 3x + 75 = 18$

$\boxed{-\ 75} \rightarrow 3x = -57$

$\boxed{\div\ 3} \rightarrow x = -19$

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Example 4 $7x - 2y = -20 \qquad 3x = 6 - 4y$

1. Bad news! Look at that 2nd equation! Might just have to add 4y to both sides to sort that mess out!

2. Let's write the second equation underneath the first…

3. Okay, bad news. We don't have the same number of either unknown. No problem, though! Why not make the number of y's the same by… multiplying ① by **2.** The signs will be different, but who cares?

4. Put a box around our Key Letters, and their signs:

5. The signs of our Key Letters are different (- and +) so we must Add equation ② to equation ①.

6. Our Key Letters have cancelled out, leaving us with a nice looking equation: $17x = -34$

7. Solve it (be careful with negatives!)

8. Use this value in one of the original equations (I'll chose ②) to find the value of the other unknown letter:

9. And now we have our two answers: $x = -2 \quad y = 3$

But we may as well check them using equation ①

① $7x - 2y = -20$ $\boxed{\begin{array}{l} x = -2 \\ y = 3 \end{array}} \rightarrow -14 - 6 = -20$

① $7x - 2y = -20$

② $3x + 4y = 6$

① × 2 → $14x - 4y = -40$

$$\begin{array}{rrrl} & ① \ 14x & \boxed{-4y} & = -40 \\ + & ② \ 3x & \boxed{+4y} & = 6 \\ \hline & 17x & & = -34 \end{array}$$

$\boxed{\div 17} \rightarrow x = -2$

② $3x + 4y = 6$

$\boxed{x = -2} \rightarrow -6 + 4y = 6$

$\boxed{+6} \rightarrow 4y = 12$

$\boxed{\div 4} \rightarrow y = 3$

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Quadratics and Simultaneous Equations?

One other thing you might be asked to do is to solve a pair of simultaneous equations where one of them is a quadratic!

Here is how I do these ones...

How Mr Barton Deal With Quadratics

1. Re-arrange your linear equation so that it is $y =$ or $x =$

2. Substitute the linear into the quadratic, being really careful about squares and negatives!

3. Your quadratic should now only have one unknown letter in it (hopefully!). So rearrange it into a nice form and then solve it by either factorising, or using the Quadratic Formula.

Remember: You will get TWO pairs of answers!

4. Use each of your answers to substitute back into one of the original equations to find TWO values for the other unknown letter.

5. Check each of these values are correct by subbing into the other original equation

Example 1 $y = x^2$ $y = 2x + 3$

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1. Good news, the linear equation ② is already in a very nice form

2. Let's substitute ② into our quadratic expression

Remember: this means that every time we see **y** in equation ①
we must replace it with 2x + 3

3. Okay, so now we have our quadratic, with only one unknown letter in it (x), instead of two!

Let's re-arrange to make it equal to zero

Now, let's cross our fingers and hope it factorises... YES!

We solve these in the usual way to get: $x = -1$ and 3

4. Now it's time to substitute each of our answers into one of the original equations (I choose ②) to find our two values for y, which gives us our answers: $y = 1$ and 9

5. But let's check these by subbing into equation ①

$\boxed{\begin{array}{l} x = -1 \\ y = 1 \end{array}}$ $y = x^2 \rightarrow 1 = (-1)^2 \rightarrow 1 = 1$

$\boxed{\begin{array}{l} x = 3 \\ y = 9 \end{array}}$ $y = x^2 \rightarrow 9 = (-3)^2 \rightarrow 9 = 9$

① $y = x^2$

② $y = 2x + 3$

② → ① $2x + 3 = x^2$

$\boxed{-2x} \rightarrow 3 = x^2 - 2x$

$\boxed{-3} \rightarrow 0 = x^2 - 2x - 3$

$(x + 1)(x - 3) = 0$

$x = -1 \qquad x = 3$

$\boxed{x = -1} \rightarrow y = -2 + 3$

$\rightarrow y = 1$

$\boxed{x = 3} \rightarrow y = 6 + 3$

$\rightarrow y = 9$

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9. Inequalities

What are Inequalities?

Inequalities are just another time-saving device invented by lazy mathematicians.
They are a way of representing massive groups of numbers with just a couple of numbers and a fancy looking symbol.

Good News: So long as you can solve equations and draw graphs, you already have all the skills you need to become an expert on inequalities!

1. What those funny looking symbols mean

$<$ means "is less than"

$\leq$ means "is less than or equal to"

$>$ means "is greater than"

$\geq$ means "is greater than or equal to"

For Example:

1.	$x < 5$	Means x is less than 5	So x could be 4, 0.6, -23... but NOT 5!
2.	$p \geq 100$	Means p is greater than or equal to 100	So p could be 104, 10000, 201.5... AND 100!
3.	$m > -2$	Means x is greater than -2	So x could be -1.9, 0, 4.3... but NOT -2!

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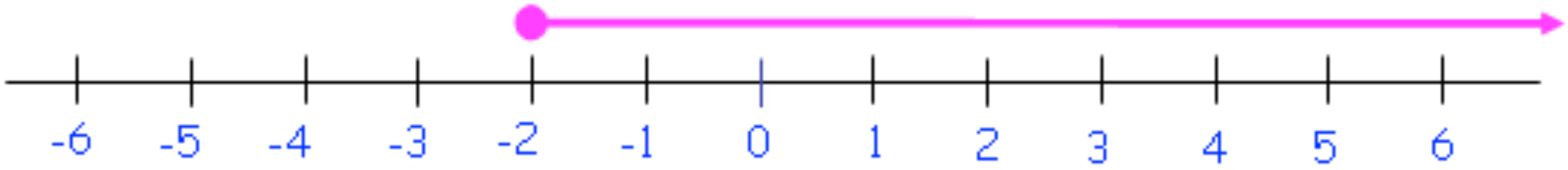
2. Representing Inequalities on a Number line

These are very common questions, and pretty easy ones too.

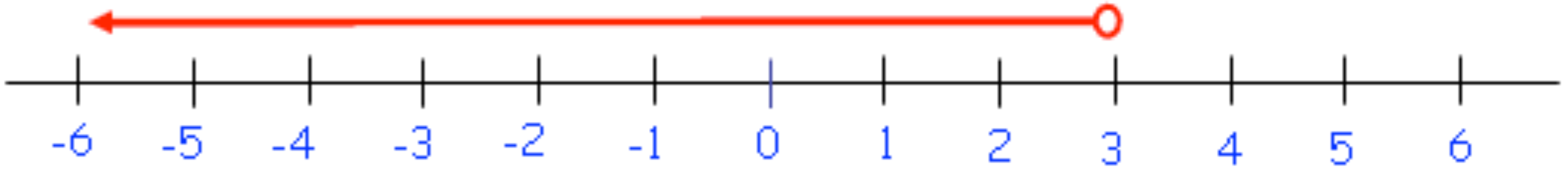
Method:

- Draw a line over all the numbers for which the inequality is true (the ones you can see, anyway)
- At the end of these lines, draw a circle, and colour it in if the inequality can equal the number, and leave it blank if it cannot.

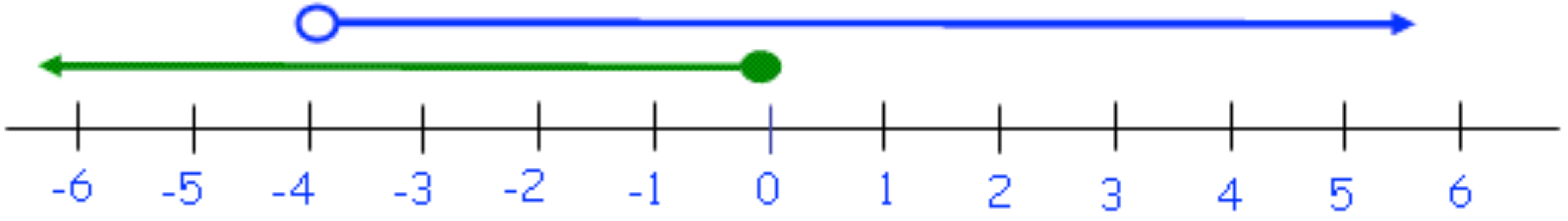
$x \geq -2$



$x < 3$



$x > -4$ and $x \leq 0$



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3. Solving Linear Inequalities

Good News: The rule for solving linear inequalities is exactly the same as that for solving linear equations – whatever you do to one side of the inequality, do exactly the same to the other

Just one thing: if you multiply or divide by a NEGATIVE, the inequality sign swaps around!

Why on earth does the sign swap around?

Imagine you have the inequality that says: 8 is greater than 5: $8 > 5$

Let's multiply both sides by 4... $\boxed{\times 4} \rightarrow$ $32 > 20$ which is still true!

Now, let's divide both sides by -2... $\boxed{\div -2} \rightarrow$ $-16 > -10$ which is NOT true!

And the only way to make the inequality true is to switch the sign around!... $-16 < -10$

Example 1 $6x + 3 \geq 27$

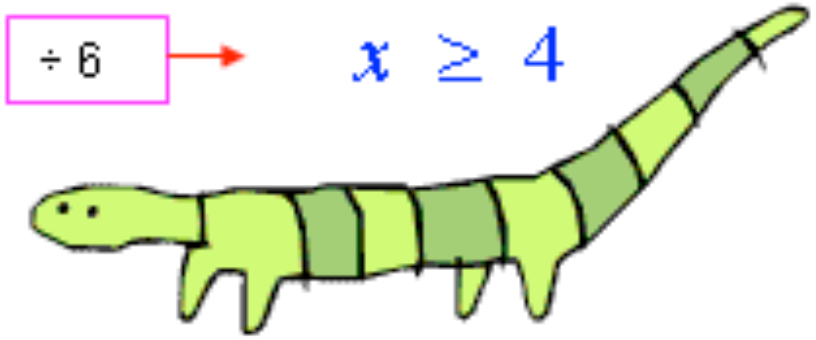
1. Okay, so just like when solving equations, we unwrap our unknown letter, by thinking about what was the last thing done to it, and doing the inverse to both sides!

2. Just need to divide both sides by 6, and we have our answer... and because 6 is positive, no need to swap any signs around!

$$6x + 3 \geq 27$$

$$\boxed{-3} \rightarrow 6x \geq 24$$

$$\boxed{\div 6} \rightarrow x \geq 4$$



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Example 2 $5x - 6 < 2x + 9$

1. Again, we do exactly the same as we would if this was an equation. Start by collecting all your x's on the side which starts off with the most x's!

2. Now we have a nice easy inequality to unwrap!

3. Which gives us our answer

$$5x - 6 < 2x + 9$$

$$\boxed{-2x} \rightarrow 3x - 6 < 9$$

$$\boxed{+6} \rightarrow 3x < 15$$

$$\boxed{\div 3} \rightarrow x < 5$$

Example 3 $-2(5x - 4) > 98$

1. Let's get those brackets expanded, being extremely careful with our negative numbers!

2. Now we begin to unwrap!

3. Notice here that we are dividing by a negative number, and so we must make sure we remember to switch our inequality sign around!

$$-2(5x - 4) > 98$$

$$\rightarrow -10x + 8 > 98$$

$$\boxed{-8} \rightarrow -10x > 90$$

$$\boxed{\div -10} \rightarrow x < -9$$

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Example 4 $-4 < 3x + 5 \leq 8$

1. This looks complicated, but all you are trying to do is unwrap the unknown letter in the middle, and whatever you do the middle, you must also do to both ends!

2. Careful unwrapping gives us our answer:

3. But what does that mean?... Well, it may become clearer when written like this:

So, x must be greater than -3 and less than or equal to 1, so x must be between -3 and 1!

$$-4 < 3x + 5 \leq 8$$

$$\boxed{-5} \rightarrow -9 < 3x \leq 3$$

$$\boxed{\div 3} \rightarrow -3 < x \leq 1$$

$x > -3$ and $x \leq 1$

4. Solving Linear Inequalities Graphically

The examiners love asking these ones, and my pupils hate doing them!

Basically, you are given one or more inequalities and you are asked to show the region on a graph which satisfies them all (i.e. every inequality works for every single point in your region)

Now, before we go on, I am going to assume you are an expert on drawing straight line graphs. If this is not the case, read Graphs 1. Straight Line Graphs before carrying on...

Method

1. Pretend the inequality sign is an equals sign and just draw your line

2. Look at the inequality sign and decide whether your line is dashed or solid

3. Pick a co-ordinate on either side of the line to help decide which region you want

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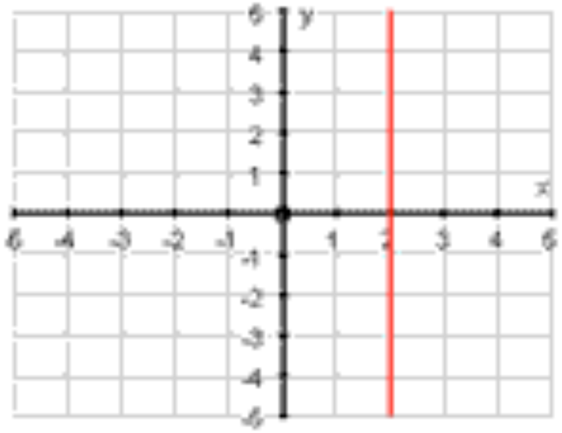
e.g. 1 $x \leq 2$

1. Draw the line x = 2



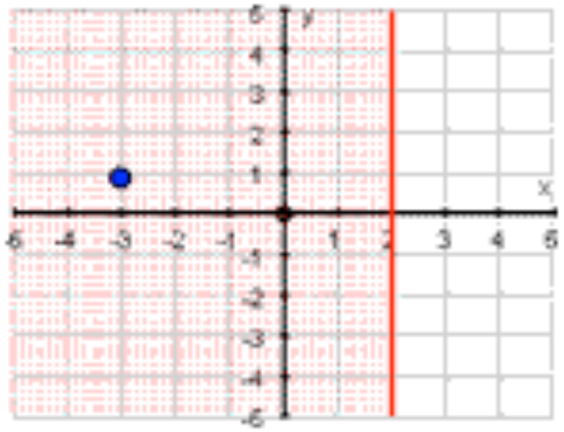
2. Notice it is a solid line as x CAN be 2

3. Choose a co-ordinate on one side of the line: e.g. (-3, 1).

$x = -3$ $y = 1$

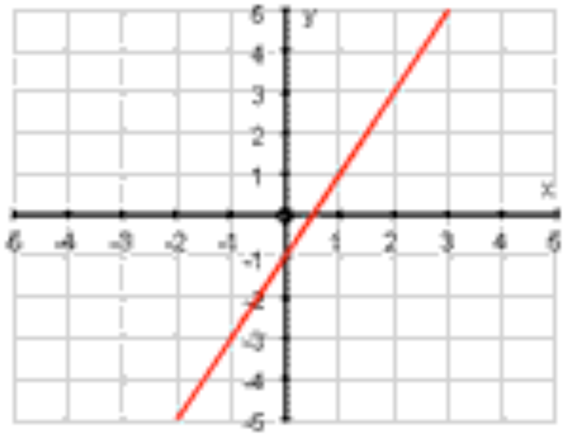
$x \leq 2 \rightarrow -3 \leq 2$ √

So our point is on the side of the line we want!



e.g. 2 $y > 2x - 1$

1. Draw the line y = 2x - 1



2. Notice it is a dashed line as y CANNOT be 2x - 1

3. Choose a co-ordinate on one side of the line: e.g. (4, 2).

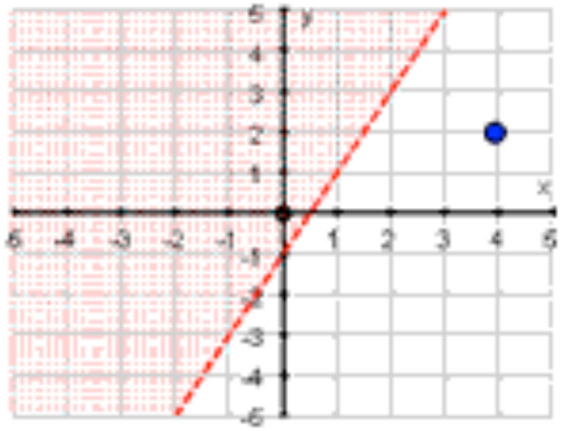
$x = 4$ $y = 2$

$y > 2x - 1$

$\rightarrow 4 > 8 - 1$

$\rightarrow 4 > 7$ ✗

So we want the other side of the line!



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e.g. 3 $x \geq 1$ $y > 2$ $5x+8y \leq 40$

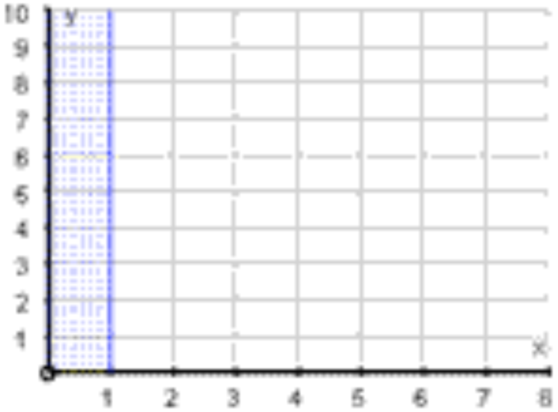
For questions like this, just deal with each inequality in turn, shading as you go!

Note: When you have got more than one inequality like this, it's normally best to shade the region you DON'T WANT, so you can leave the region you do want blank!

$x \geq 1$

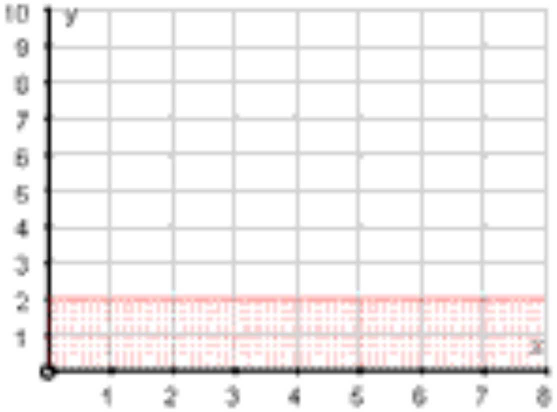
You should be able to do this one all in one.

The points where x is greater than 2 are to the right, so shade the left!



$y > 2$

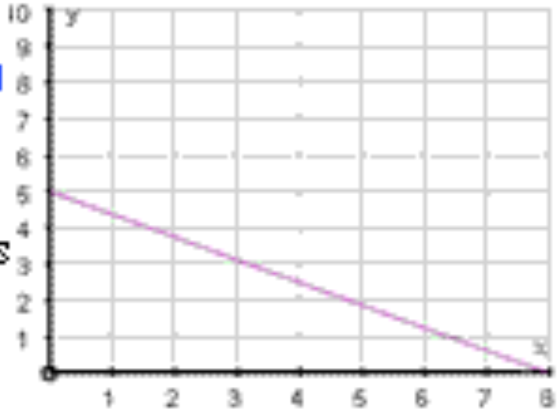
The big y values are all above the line, so let's shade the ones we don't want below the line!



$5x+8y \leq 40$

1. Draw the line 5x+8y = 40

2. Notice it is a solid line as 5x+8y CAN be equal to 40



3. Choose a co-ordinate on one side of the line: e.g. (2, 1).

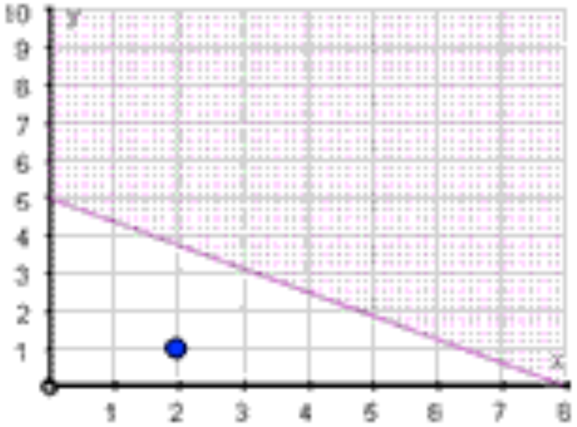
$x = 2$ $y = 1$

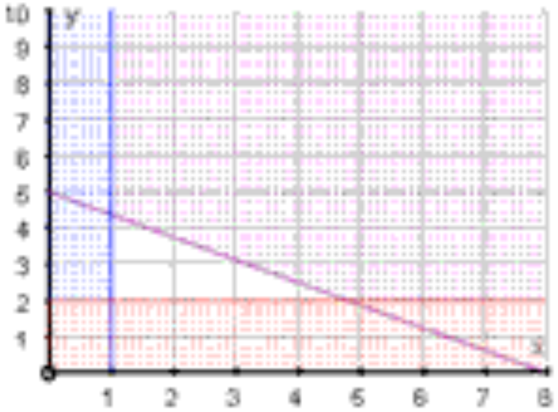
$5x + 8y < 40$

→ $10 + 8 < 40$

→ $18 > 7$ √

So we shade the other side





Putting it all together leaves us the blank region in the middle that satisfies all the inequalities!

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5. Solving Quadratic Inequalities – warning, these are hard!

Now, I have a way of doing these which may be different to how you have been taught, so feel free to completely ignore my method... but I must admit I think it's pretty good!

How Mr Barton Solves Quadratic Inequalities

1. Do the same to both sides to make the quadratic inequality as simple as possible

2. Sketch the simplified quadratic inequality

3. Use the sketch to find the values which satisfy the inequality

Example $2x^2 + 3 \geq 53$

1. Okay, so let's use our algebra skills to get this quadratic inequality as simple as possible:

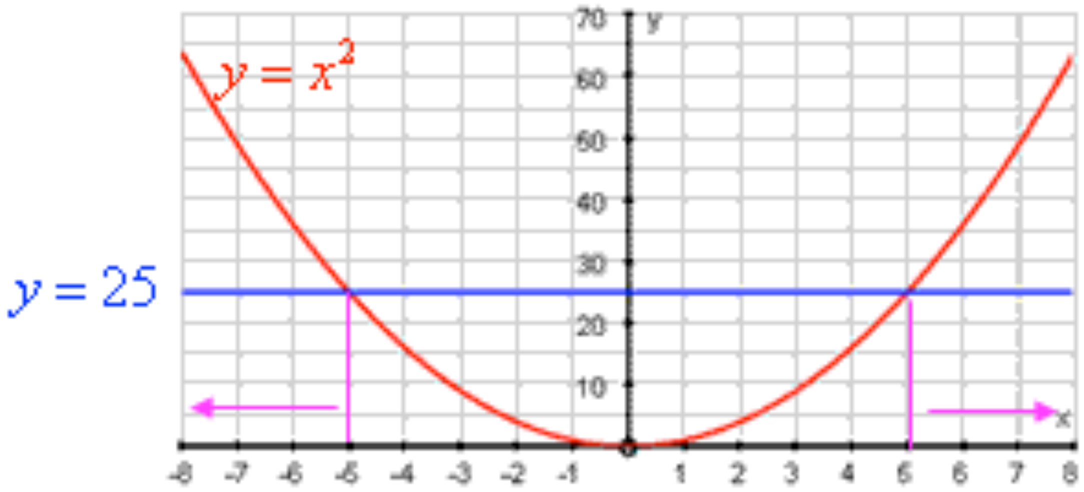
$$2x^2 + 3 \geq 53$$

$$\boxed{-3} \rightarrow 2x^2 \geq 50$$

$$\boxed{\div 2} \rightarrow x^2 \geq 25$$

2. Now, let's think about what this inequality is saying: "we want all the values of x where x^2 is greater than our equal to 25"

Let's sketch that!



Now all we need to ask ourselves is: "for what values of x is x^2 bigger than 25?"... Well, from our graph it looks like the answer is when x is either bigger than 5 or smaller than -5, which gives our answer:

$$x \geq 5 \quad \text{or} \quad x \leq -5$$

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10. Indices

What are Indices?

Indices are just a fancy word for "power"

They are the little numbers or letters that float happily in the air next to a number or letter

A bit of indices lingo:

The index or power

The base

$$5^4$$

Two things you must remember about indices...

1. Indices only apply to the number or letter they are to the right of - the base

e.g. in abc^2, the squared only applies to the c, and nothing else. If you wanted the squared to apply to each term, it would need to be written as $(abc)^2$.

2. Indices definitely do not mean multiply

e.g. 6^3 definitely does not mean 6 × 3, it means 6 × 6 × 6!

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Rule 1 – The Multiplication Rule

Using fancy notation: $a^m \times a^n = a^{m+n}$

What it actually means: Whenever you are multiplying two terms with the same base, you can just add the powers!

Numbers: If there are numbers IN FRONT of your bases, then you must multiply those numbers together as normal

Examples

$x^3 \times x^4 = x^7$ √ Classic wrong answer: x^{12} ×

$2^5 \times 2^3 = 2^8$ √ Classic wrong answer: 4^8 ×

$3p^4 \times 2p^5 = 6p^9$ √ Classic wrong answer: $6p^{20}$ ×

$2ab^2c \times 5ab^2c^3 = 10a^2b^4c^4$ √

Remember: if a base does not appear to have a power, the power is a disguised 1!

e.g. $2ab^2c = 2a^1b^2c^1$

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Rule 2 – The Division Rule

Using fancy notation: $a^m \div a^n = a^{m-n}$ Or $\frac{a^m}{a^n} = a^{m-n}$

What it actually means: Whenever you are dividing two terms with the same base, you can just subtract the powers!

Numbers: If there are numbers IN FRONT of your bases, then you must divide those numbers as normal

Examples

$x^{12} \div x^4 = x^8$ √ Classic wrong answer: x^3 ×

$\frac{5^7}{5^3} = 5^4$ √ Classic wrong answer: 1^4 ×

$\frac{20k^{10}}{5k^5} = 4k^5$ √ Classic wrong answer: $4k^2$ ×

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Rule 3 – The Power of a Power Rule

Using fancy notation: $(a^m)^n = a^{m \times n}$

What it actually means: Whenever you have a base and it's power raised to another power, you simply multiply the powers together but keep the base the same!

Numbers: If there is a number IN FRONT of your base, then you must raise that number to the power

Examples

$(x^5)^3 = x^{15}$ √ Classic wrong answer: x^8 ×

$(2^3)^2 = 2^6$ √ Classic wrong answer: 4^6 ×

$(3a^4)^3 = 27a^{12}$ √ Classic wrong answer: $9a^{12}$ ×

$(2a^3b^2c)^5 = 32a^{15}b^{10}c^5$ √

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Examples Using all Three Rules

Rule 1: $a^m \times a^n = a^{m+n}$

Rule 2: $\dfrac{a^m}{a^n} = a^{m-n}$

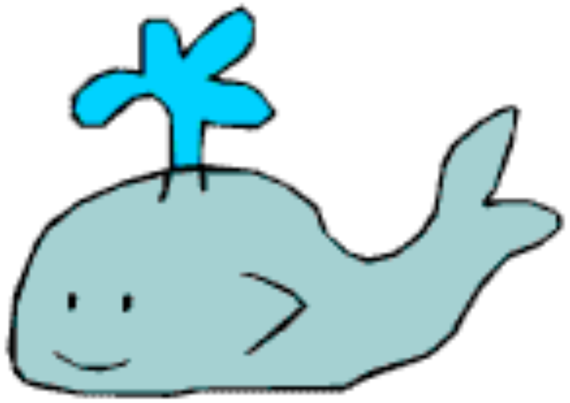
Rule 3: $(a^m)^n = a^{m \times n}$

1. $$\frac{x^3 \times (x^2)^4}{x^5} \xrightarrow{\text{Rule 3}} \frac{x^3 \times x^8}{x^5} \xrightarrow{\text{Rule 1}} \frac{x^{11}}{x^5} \xrightarrow{\text{Rule 2}} x^6$$

2. $$\frac{(5^3)^2 \times (5^2)^{10}}{(5^5)^2 \times 5} \xrightarrow{\text{Rule 3}} \frac{5^6 \times 5^{20}}{5^{10} \times 5^1} \xrightarrow{\text{Rule 1}} \frac{5^{26}}{5^{11}} \xrightarrow{\text{Rule 2}} x^{15}$$

3. $$\frac{(5v^4)^2 \times (2v^5)^4}{50v} \xrightarrow{\text{Rule 3}} \frac{25v^8 \times 16v^{20}}{50v} \xrightarrow{\text{Rule 1}} \frac{400v^{28}}{50v^1}$$

$$\xrightarrow{\text{Rule 2}} 8v^{27}$$



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Rule 4 – The Zero Index

Using fancy notation: $a^0 = 1$

What it actually means: Anything to the power of zero is 1!

Examples $x^0 = 1$ $17^0 = 1$ $5x^0 = 5 \times 1 = 5$

Rule 5 – Negative Indices

Using fancy notation: $a^{-m} = \frac{1}{a^m}$

What it actually means: A negative sign in front of a power is the same as writing "one divided by the base and power". The posh name for this is the RECIPROCAL

Watch out! Only the power and base are flipped over, nothing else!

Examples $x^{-2} = \frac{1}{x^2}$ $5^{-4} = \frac{1}{5^4}$ $5a^{-3} = \frac{5}{a^3}$

$$\left(\frac{1}{3}\right)^{-1} = \left(\frac{3}{1}\right)^{1} = 3 \qquad \left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^{2} = 16 \qquad \left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^{3} = \frac{27}{8}$$

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Rule 6 – Fractional Indices

Using fancy notation: $a^{\frac{1}{n}} = \sqrt[n]{a}$

What it actually means: When a power is a fraction it means you take the root of the base… and which root you take depends on the number on the bottom of the fraction!

The main ones:

$a^{\frac{1}{2}} = \sqrt{a}$ The power of a half means take the square-root!

$a^{\frac{1}{3}} = \sqrt[3]{a}$ The power of a third means take the cube-root!

Examples

$64^{\frac{1}{2}} = \sqrt{64} = 8$

$27^{\frac{1}{3}} = \sqrt[3]{27} = 3$ Because $3^3 = 27$

$32^{\frac{1}{5}} = \sqrt[5]{32} = 2$ Because $2^5 = 32$

For ones like the last two it is worth learning your powers of 2 and 3:

$2^2 = 4$ $3^2 = 9$

$2^3 = 8$ $3^3 = 27$

$2^4 = 16$ $3^4 = 81$

$2^5 = 32$

$2^6 = 64$

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Flip It, Root It, Power It!

Sometimes you get asked some indices questions that look an absolute nightmare, but if you just deal with each aspect in turn, then you will be fine:

1. Flip It - If there is a negative sign in front of your power, flip the base over and we're positive!

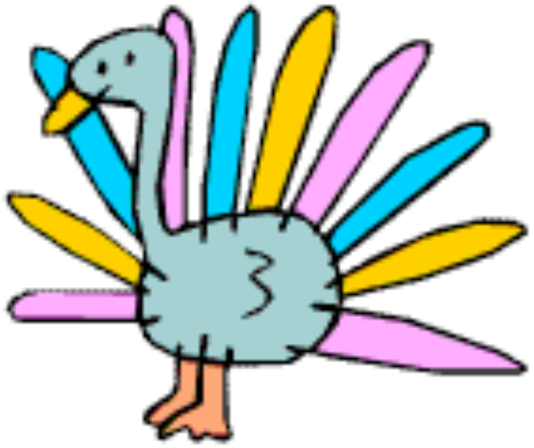
2. Root It - If your power is a fraction, then deal with the bottom of it by rooting your base

3. Power It - When all that is sorted, just raise your base to the remaining power and you're done!

Examples

1. $8^{-\frac{2}{3}} \xrightarrow{\text{Flip it}} \left(\frac{1}{8}\right)^{\frac{2}{3}} \xrightarrow{\text{Root It}} \left(\frac{\sqrt[3]{1}}{\sqrt[3]{8}}\right)^{2} = \left(\frac{1}{2}\right)^{2}$

$$\xrightarrow{\text{Power It}} \frac{1^2}{2^2} = \frac{1}{4}$$



2. $\left(\frac{1}{64}\right)^{-\frac{5}{6}} \xrightarrow{\text{Flip it}} 64^{\frac{5}{6}} \xrightarrow{\text{Root It}} \left(\sqrt[6]{64}\right)^{5} = 2^{5}$

$$\xrightarrow{\text{Power It}} 32$$

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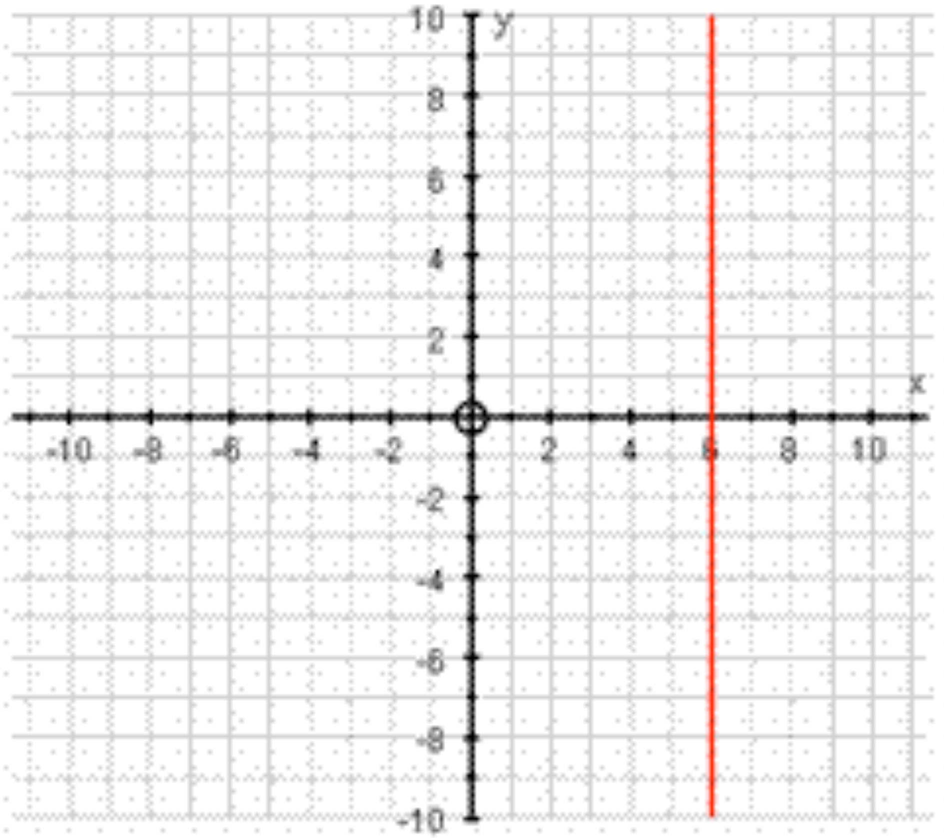
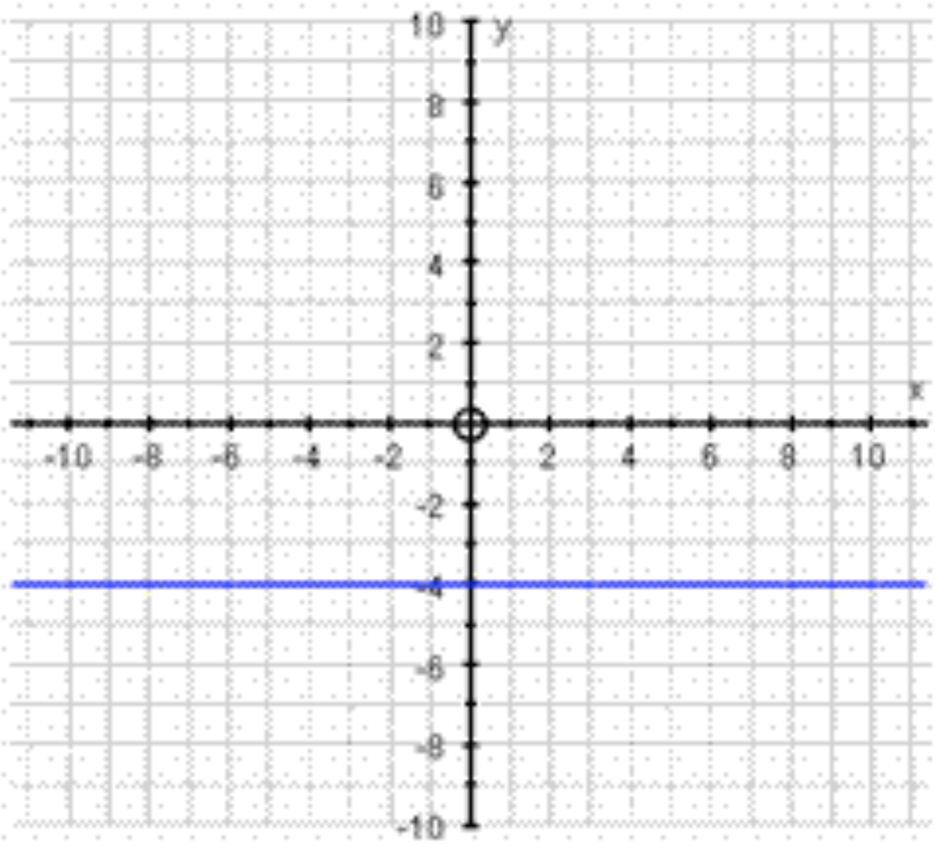
1. Straight Line Graphs

1. The ones you should know, but which everyone mixes up

You need to learn how to recognise and draw horizontal and vertical lines.
I have put two examples below... have a look at them and get them fixed into your brain!



Every single point on this line has an x co-ordinate of 6, so the equation of the line is: **x = 6**



Every single point on this line has a y co-ordinate of -4, so the equation of the line is: **y = -4**

Note: The equation of the x axis is **y = 0**... and the equation of the y axis is **x = 0**!

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2. What does the Equation of a Straight Line actually mean?

The equation of a straight line is just a way of writing the relationship between the x co-ordinates and the y co-ordinates that lie on that line.

Example: **$y = 2x - 1$**

This says that the relationship between all the x co-ordinates and all the y co-ordinates is:
"get your x co-ordinate, multiply it by 2, subtract 1, and you get your y co-ordinate"

So...If a pair of co-ordinates has this relationship... such as (5, 9)... then it's on the line

If it doesn't... such as (3, 2)... then it does not lie on the line

What you end up with is just a straight line that goes through all the co-ordinates which share that relationship

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3. Drawing Straight Line Graphs from their Equation

As well as the horizontal and vertical lines, there are 2 other types of straight line graph equation, but they both follow the same general method:

1. Choose a sensible value of x... one that is small enough to fit on the paper, and easy enough for you to work out

2. Carefully substitute it into the equation to get your y value

3. Do this 4 times so you have four points

4. Join them up with a straight line

Crucial: If one of your points does not lie on the straight line, then I'm afraid you have made a mistake... but at least you know which one is wrong so it should be easy to fix!

Number 1 Classic Mistake People Make:

Messing up their negative numbers... you must be very careful when substituting negative x's

One Final Top Tip

Pick x = 0 as one of your points, as it is often nice and easy to work out the y value!

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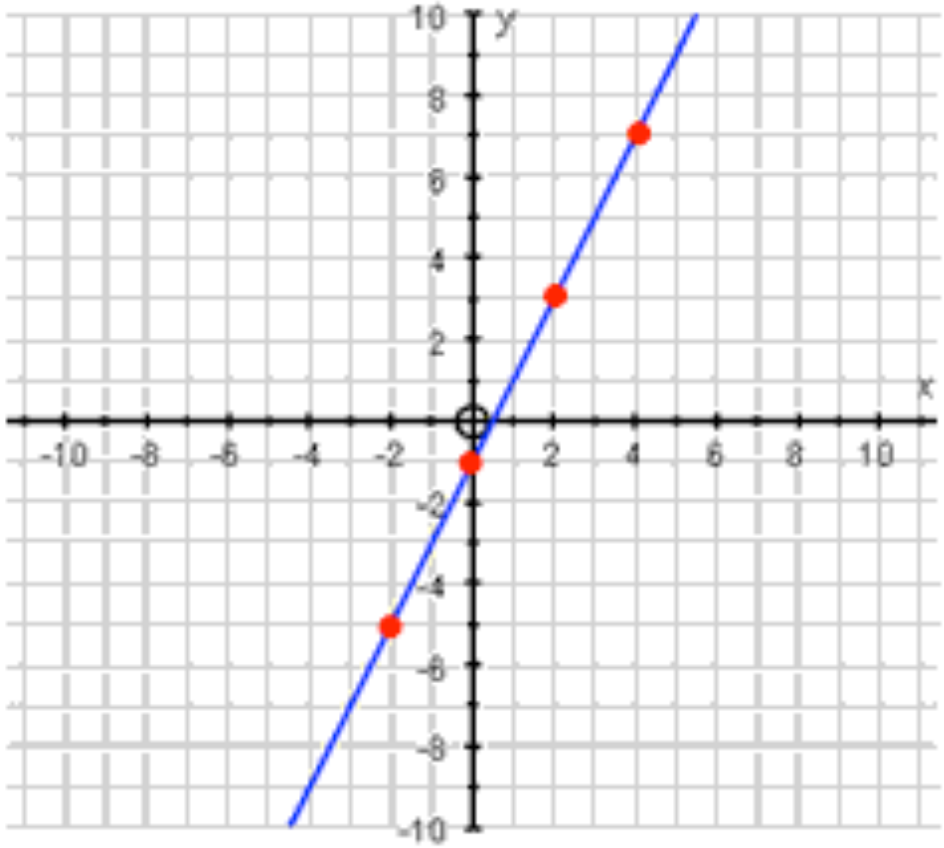
Type 1: y =

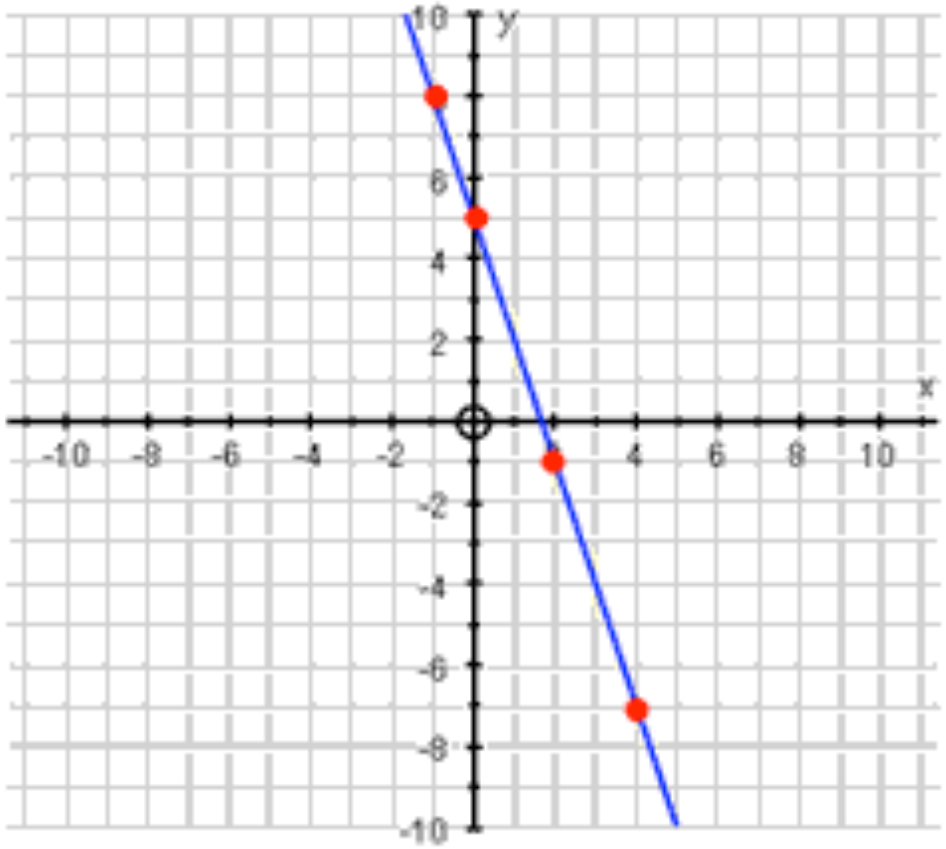
$$y = 2x - 1$$

x	0	2	4	-2
y	-1	3	7	-5



$$y = -3x + 5$$

x	0	2	4	-1
y	5	-1	-7	8



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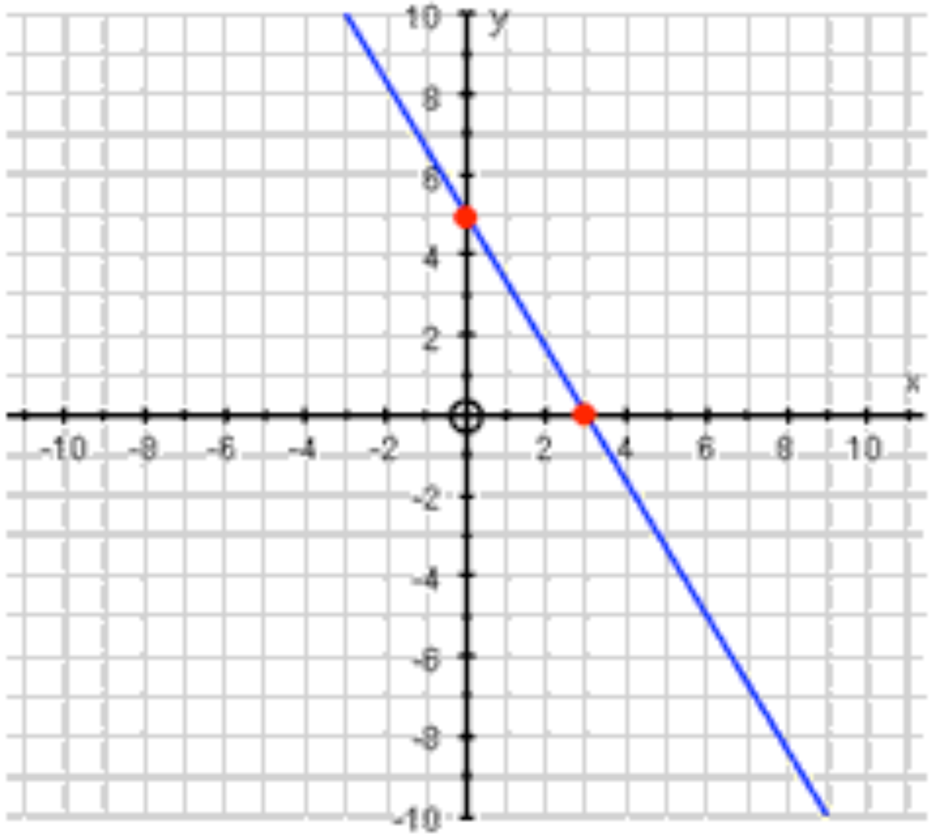
Type 2: x + y = number

The technique for these is just the same – you still want to find points that lie on the line – but unless you can spot a good pair that works, try substituting **$x = 0$** to get a y co-ordinate, and then **$y = 0$** to get an x co-ordinate!

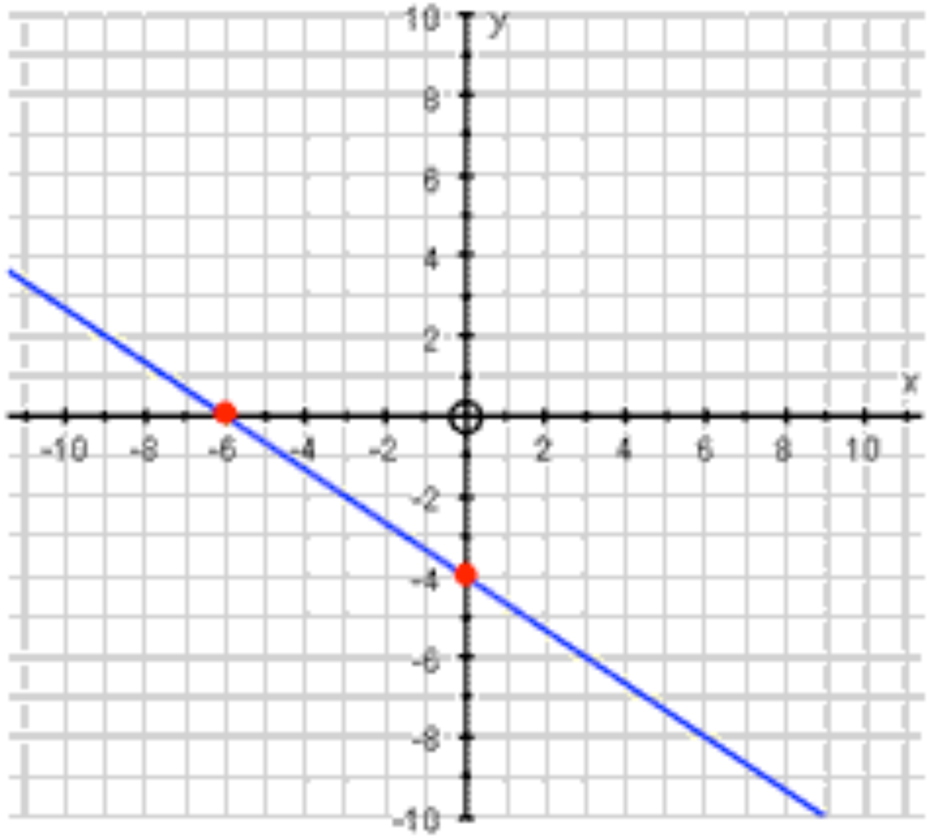
$$5x + 3y = 15$$

x	0	3
y	5	0



$$4x + 6y = -24$$

x	0	-6
y	-4	0



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4. What can we learn from the Equation of a Line: $y = mx + c$

The more Type 1 lines you draw, the more you should start noticing the link between the equation of the line, and what the line actually looks like

These are the Big Facts that you need to know!

$$y = mx + c$$

m

- This number tells you the gradient/steepness of the line
- The bigger the number, the steeper the line
- If the number is positive, the line slopes upwards
- If it is negative, the line slopes downwards
- Parallel lines have the same gradient

+ c

- This number tells you where the line crosses the y axis
- Its posh name is the y intercept

Classic Mistake: If the equation is NOT in the form: $y = mx + c$, you must first re-arrange it before you start saying what the gradient and intercept are!

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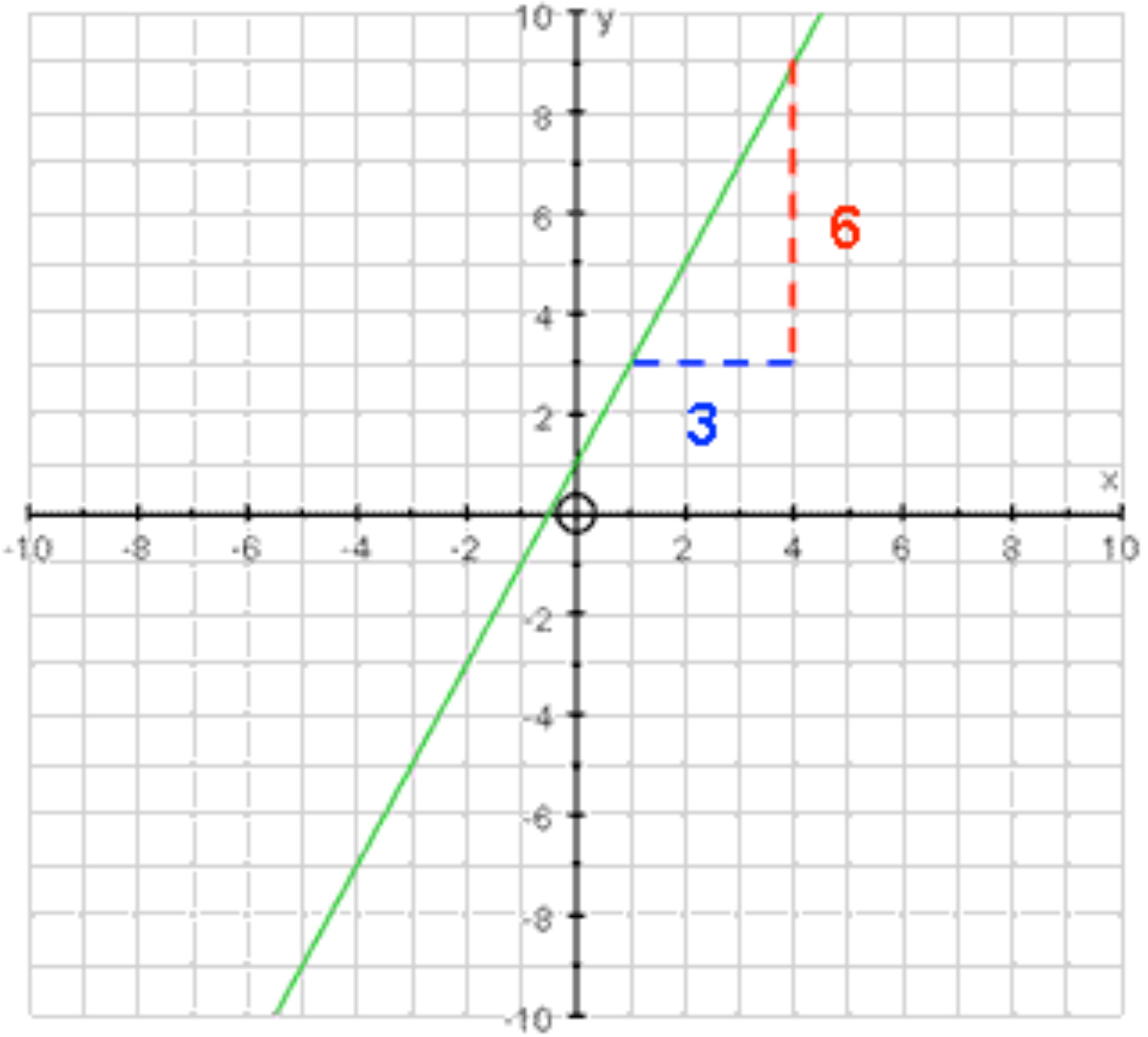
5. Working Out the Equation of a Line

Using our knowledge of $y = mx + c$, we can actually work backwards and figure out the equation of a straight line just by looking at it!



First we must work out the gradient of the line, and we do this by drawing a right-angled triangle anywhere on the line and using this lovely formula:

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x}$$

$$\text{Gradient} = \frac{6}{3} = 2$$

Now all we need is our y-intercept, which is just the place the line crosses the y axis… which is (0, 1)!

So, our equation must be… $y = 2x + 1$

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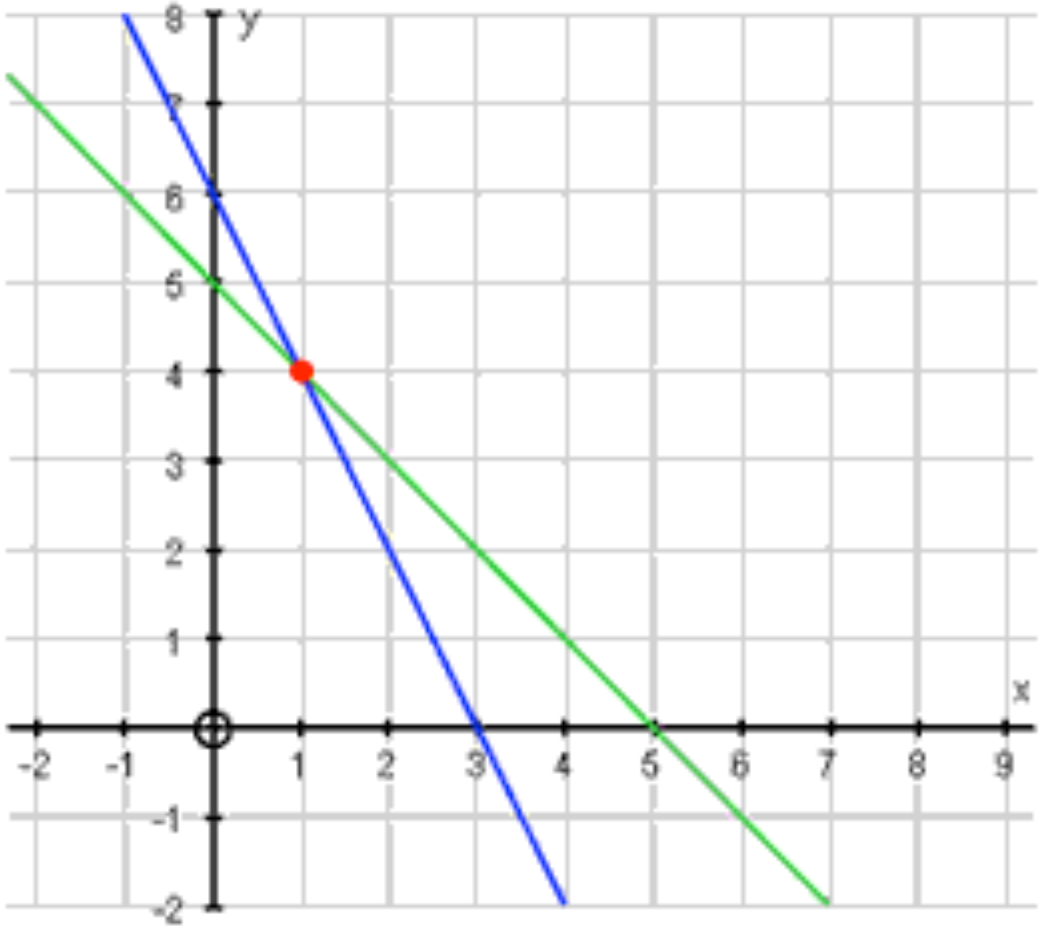
6. Using Straight Line Graphs to Solve Simultaneous Equations

As I mentioned in Algebra 8, it is possible to use straight line graphs to solve simultaneous equations

All you need to do it to carefully plot both lines, and the point where they cross is your answer… but remember you want $x =$ and $y =$



Example: Solve the following pair of simultaneous equations graphically:

$x + y = 5$ and $2x + y = 6$

$$x + y = 5$$

x	0	5
y	5	0

$$2x + y = 6$$

x	0	3
y	6	0

So, our solution must be… $x = 1$ and $y = 4$

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2. Quadratics and Cubics

1. What does the Equation of a Curve actually mean?

The equation of a curve, whether it be a quadratic, a cubic, or anything else, is just a way of expressing the relationship between the x co-ordinates and the y co-ordinates that lie on that curve.

Example: $y = x^2 + 3x - 9$

This says that the relationship between all the x co-ordinates and all the y co-ordinates is: "get your x co-ordinate, square it, add on three lots of your x co-ordinate, subtract 9, and you get your y co-ordinate"

So...If a pair of co-ordinates has this relationship... such as (2, 1)... then it's on the curve

If it doesn't... such as (5, 4)... then it does not lie on the curve

What you end up with is just a curve that goes through all the co-ordinates which share that relationship

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2. Drawing Curves from their Equation

The method is identical to how we drew straight lines

1. Choose a sensible value of x… one that is small enough to fit on the paper, and easy enough for you to work out

2. Carefully substitute it into the equation to get your y value

3. Do this enough times to see the shape of the curve

4. Join them up with a smooth curve (don't have any sharp, pointy bits)

Crucial: You are more likely to get the shape of the curve right if you have a good knowledge of what shapes different equations make! Have a quick read though 3. Shapes of Graphs before you carry on!

Number 1 Classic Mistake People Make:

Messing up their negative numbers… you must be very careful when substituting negative x's, whether you are doing this on a calculator or in your head (see the next 2 sections)

One Final Top Tip

Pick $x = 0$ as one of your points, as it is often nice and easy to work out the y value!

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3. Substituting Numbers in your Head

If you are asked to draw a curve on a non-calculator paper, then you will need to be very careful

Things to remember

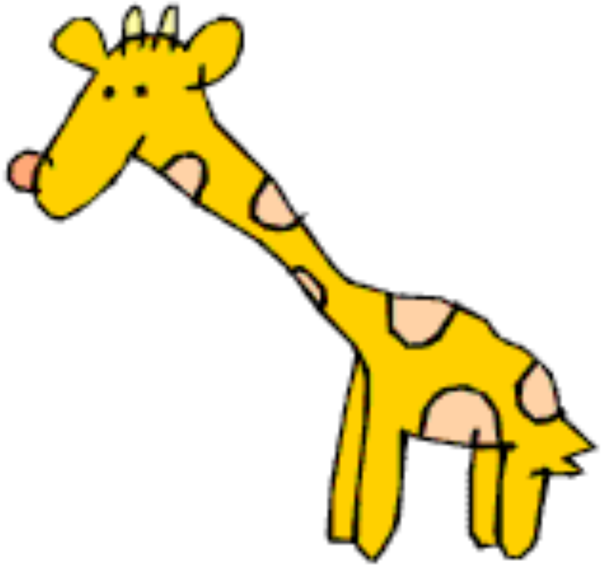
1. What order you must do operations - remember BODMAS??

2. All you rules of negative numbers!

Example

If I was trying to substitute $x = -2$ into $y = x^2 - 4x + 2$, then this is what I would be saying to myself in my head:

- Okay, let's deal with the squared term first...
- $(-2)^2$ is equal to... 4, because when you square a negative you get a positive...
- Next up is 4x... which is 4 multiplied by x...
- Which is $4 \times (-2)$...
- Which is equal to -8
- So, I have... $4 - -8 + 2$
- Well, those two minuses are touching, so they become a plus
- So I have... $4 + 8 + 2$...
- Which equals 10
- So, the point I need to plot has the co-ordinates **(-2, 10)**



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4. Substituting Numbers using a Calculator

Whilst having a calculator makes doing tricky sums much easier, it also means you are likely to get much more difficult numbers to work with, and if you are not careful, calculators can do some daft things!

Things to remember

1. Always put your negative numbers in brackets

2. Always do each calculation twice to make sure you didn't press a wrong button!

Example

If I was trying to substitute $x = -4$ into $y = x^3 + 2x^2 - 6x + 2$, then this is the order I would press the buttons:

(- 4) X^3 + 2 X (- 4) X^2

x^3 $2x^2$

- 6 X (- 4) + 2

$6x$ 2

And if you do all that, you should get a y value of... **-6**

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5. Using Curves to Solve Equations

Seeing as you have taken all that time drawing a beautiful curve, you may as well use it to solve an equation

Method

1. If it isn't already, re-arrange the equation so all the letters are on the left, and there is either a number or a zero on the right hand side
2. Draw the graph of the left hand side of the equation
3. On your graph, draw a horizontal line through whatever number was on the right hand side of your equation
4. Mark on the points where this horizontal line crosses your curve
5. The x co-ordinates of these points are the solutions to the equation

Note: If there is a zero on the right hand side of the equation, you are just looking for the points where the curve crosses the x axis!

6. Putting it all Together

What follows now are three examples of drawing graphs and then using them to solve equations

I suggest you make sure you can get each of the numbers in the table yourself... both in your head and on a calculator!

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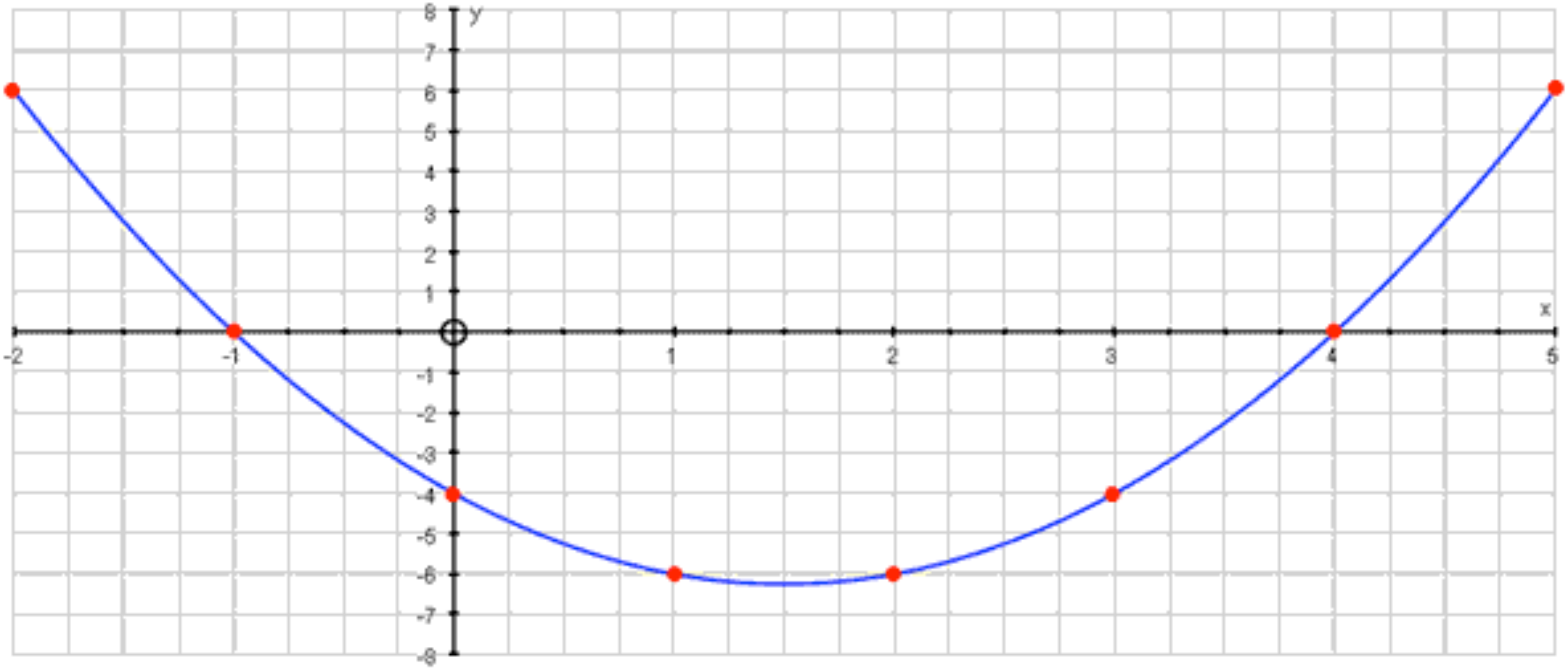
Example 1

$y = x^2 - 3x - 4$

x	-2	-1	0	1	2	3	4	5
y	6	0	-4	-6	-6	-4	0	6



Use the graph to solve: $x^2 - 3x - 4 = 0$

We are looking for where the curve crosses the x axis, which gives us solutions of:

x = -1 and x = 4

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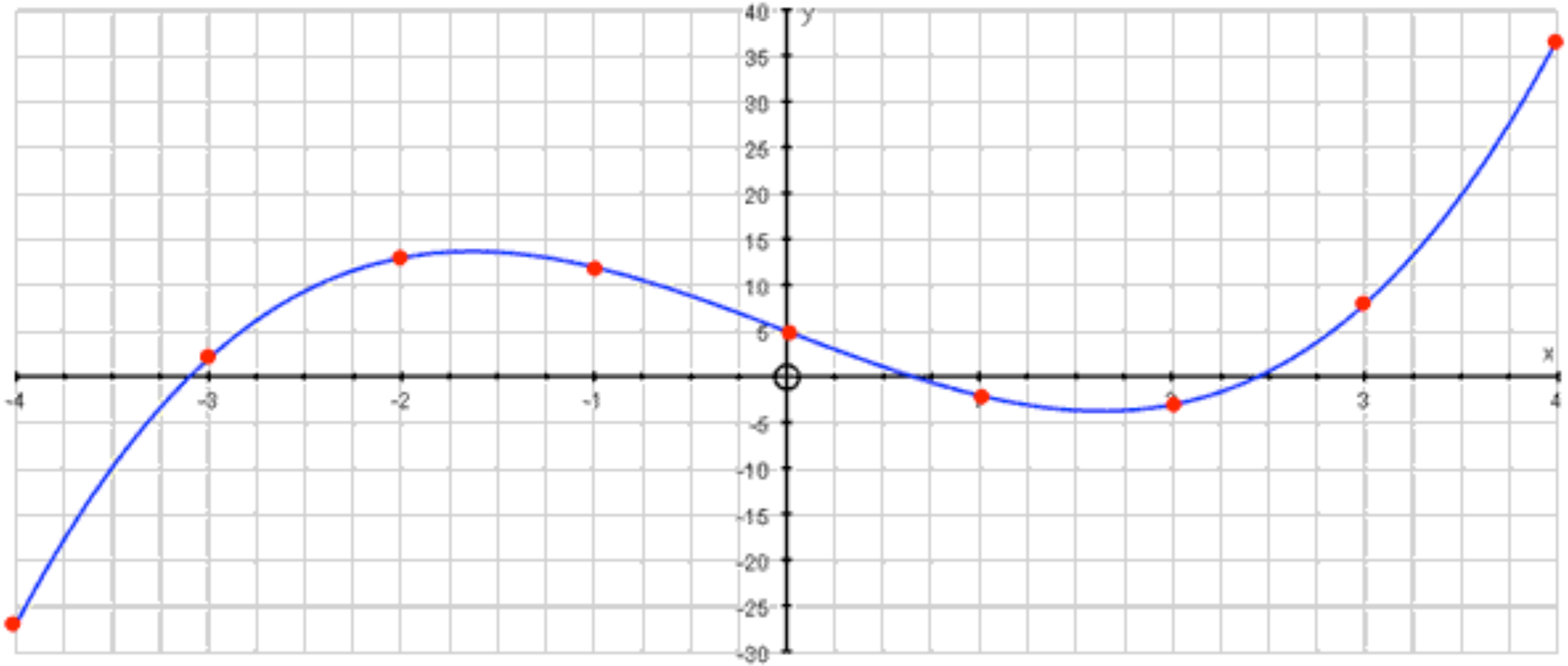
Example 2

$y = x^3 - 8x + 5$

x	-4	-3	-2	-1	0	1	2	3	4
y	-27	2	13	12	5	-2	-3	8	37



Use the graph to solve: $x^3 - 8x + 5 = 0$

Again, we are looking for where the curve crosses the x axis, which gives us solutions of:

x = -3.1 x = 0.7 and x = 2.5 ... these are only rough answers, but that doesn't matter!

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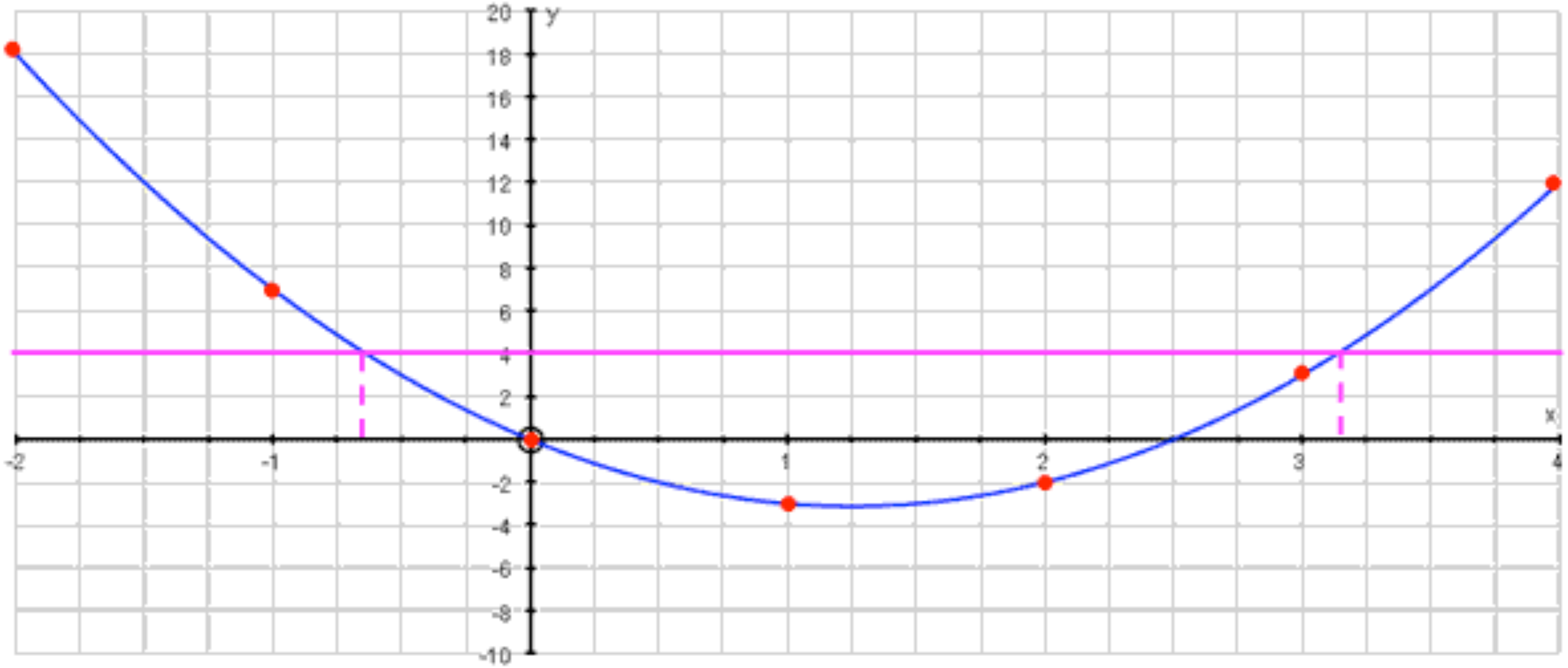
Example 3

$y = 2x^2 - 5x$

x	-2	-1	0	1	2	3	4
y	18	7	0	-3	-2	3	12



Use the graph to solve: $2x^2 - 5x = 4$

We must draw in the line y = 4 and read off the x co-ordinates of where the line hits the curve

x = -0.7 and x = 3.2

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3. Shapes of Graphs

The Importance of knowing the Shapes of Graphs...

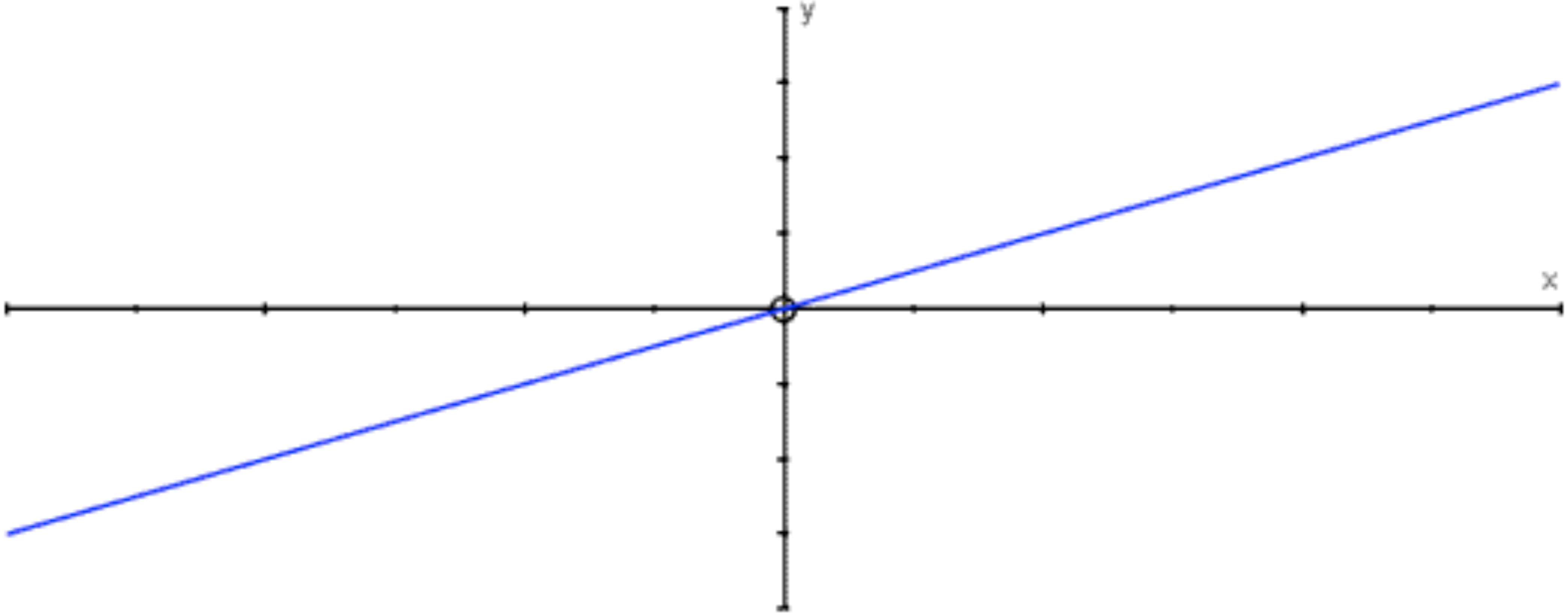
If you can look at the equation of a graph and already have a pretty good idea of what shape it will be, you are more likely to spot any silly errors when plotting it

NOTE: What follows are general shapes of graphs – learn to spot the key features of each!

1. Positive x

Equation: Highest power of x is **1**, and the x term is positive

Examples: $y = 2x + 3$ $y = x - 8$ $y = 5x$ $y = 9x - 6$



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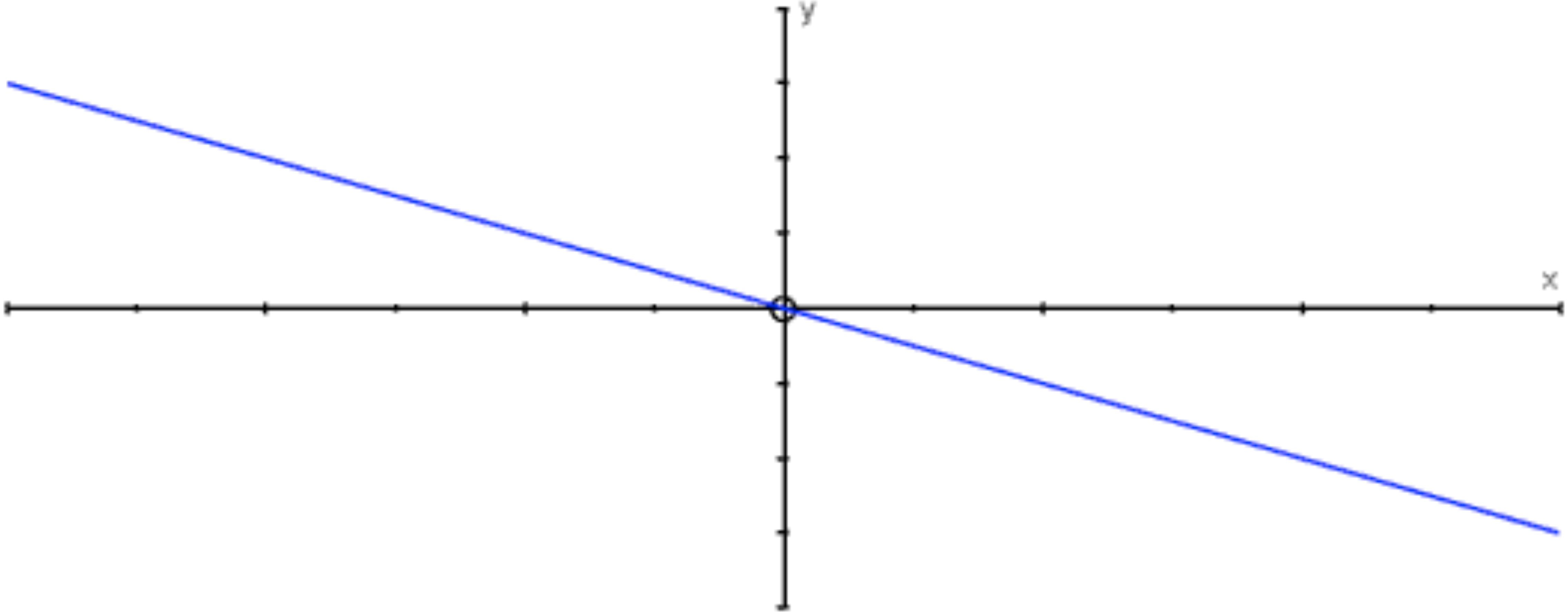
2. Negative x

Equation: Highest power of x is **1**, and the x term is negative

Examples: $y=-5x+3$ $y=-x-3$ $y=-7x$ $y=5-6x$



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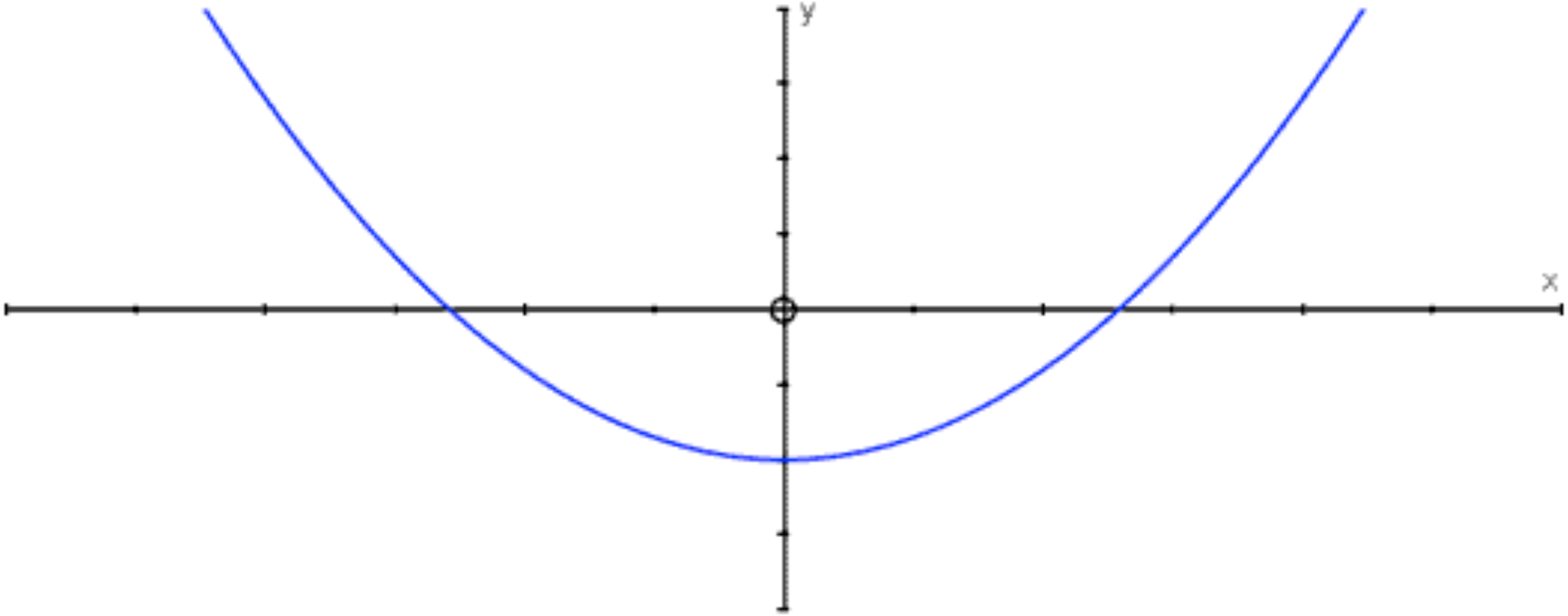
3. Positive x^2

Equation: Highest power of x is **2**, and the x^2 term is positive

Examples: $y = x^2$ $y = x^2 + 5$ $y = x^2 - 3x + 2$ $y = 3x^2 + 2x - 6$



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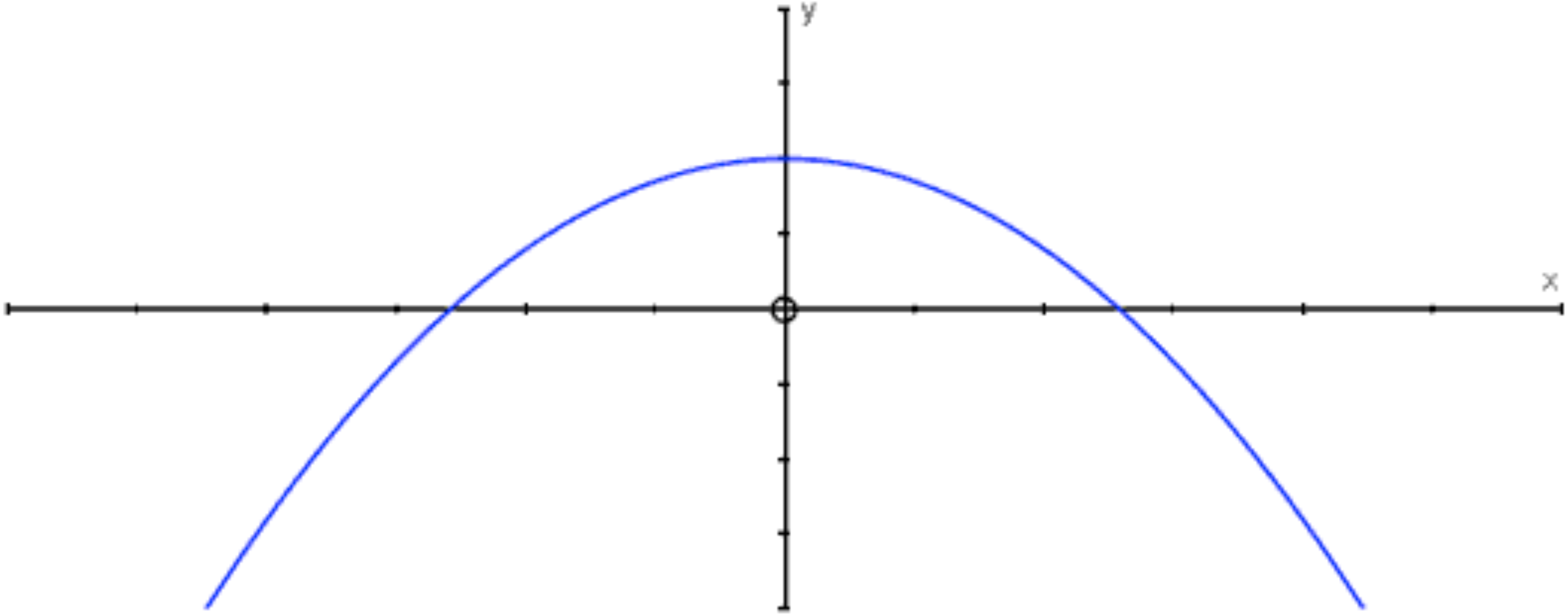
4. Negative x^2

Equation: Highest power of x is **2**, and the x^2 term is negative

Examples: $y = -x^2$ $y = -2x^2 + 4$ $y = -2(x^2 - 5x + 5)$ $y = 5 + 3x - x^2$



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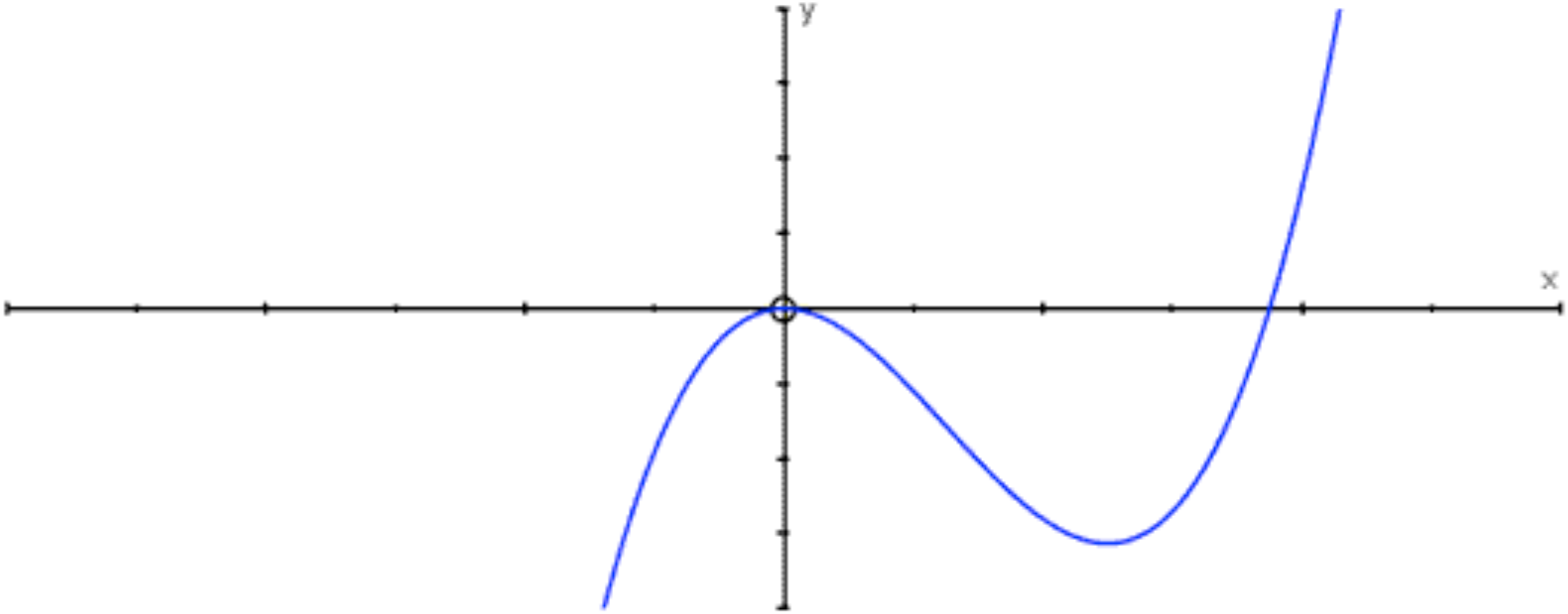
5. Positive x^3

Equation: Highest power of x is **3**, and the x^3 term is positive

Examples: $y = x^3$ $y = x^3 + 10$ $y = x^3 - 2x^2 - 4x + 2$ $y = 2x^3 - 6x$



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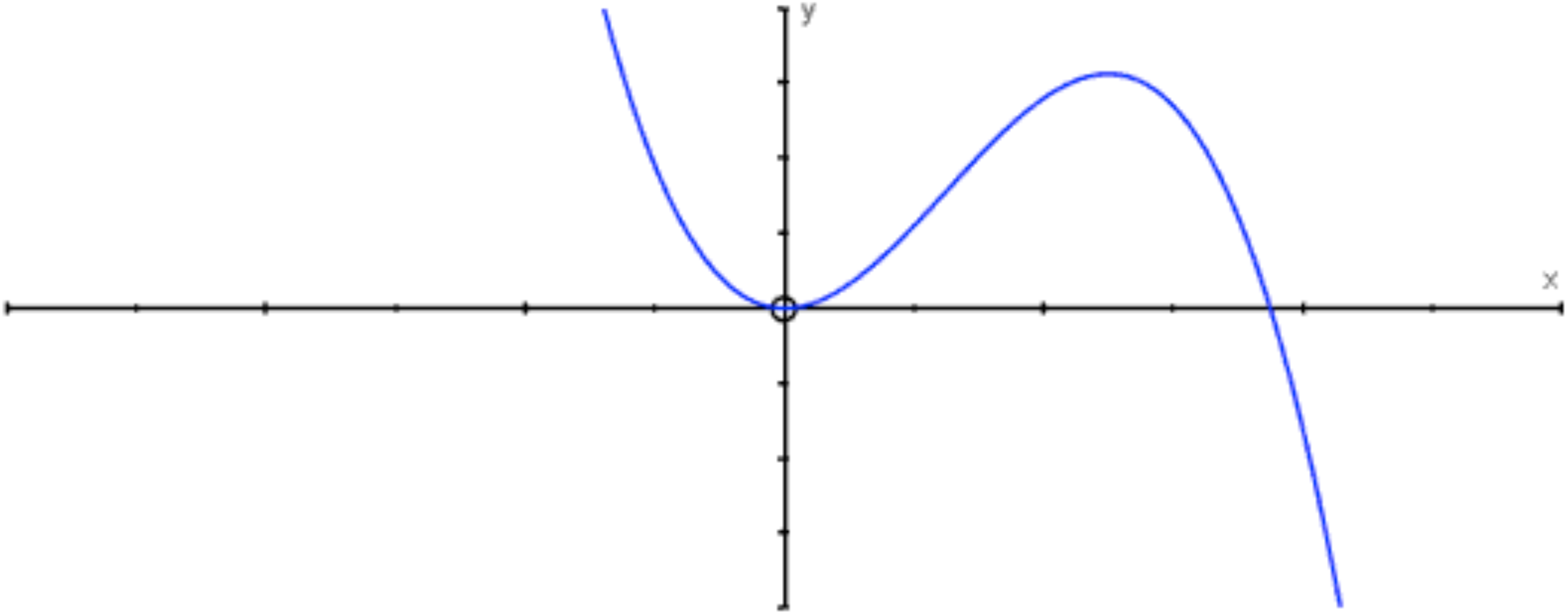
6. Negative x^3

Equation: Highest power of x is **3**, and the x^3 term is negative

Examples: $y=-x^3$ $y=-5x^3+2$ $y=-4x^3+x^2+5$ $y=5+3x+5x^2-x^3$



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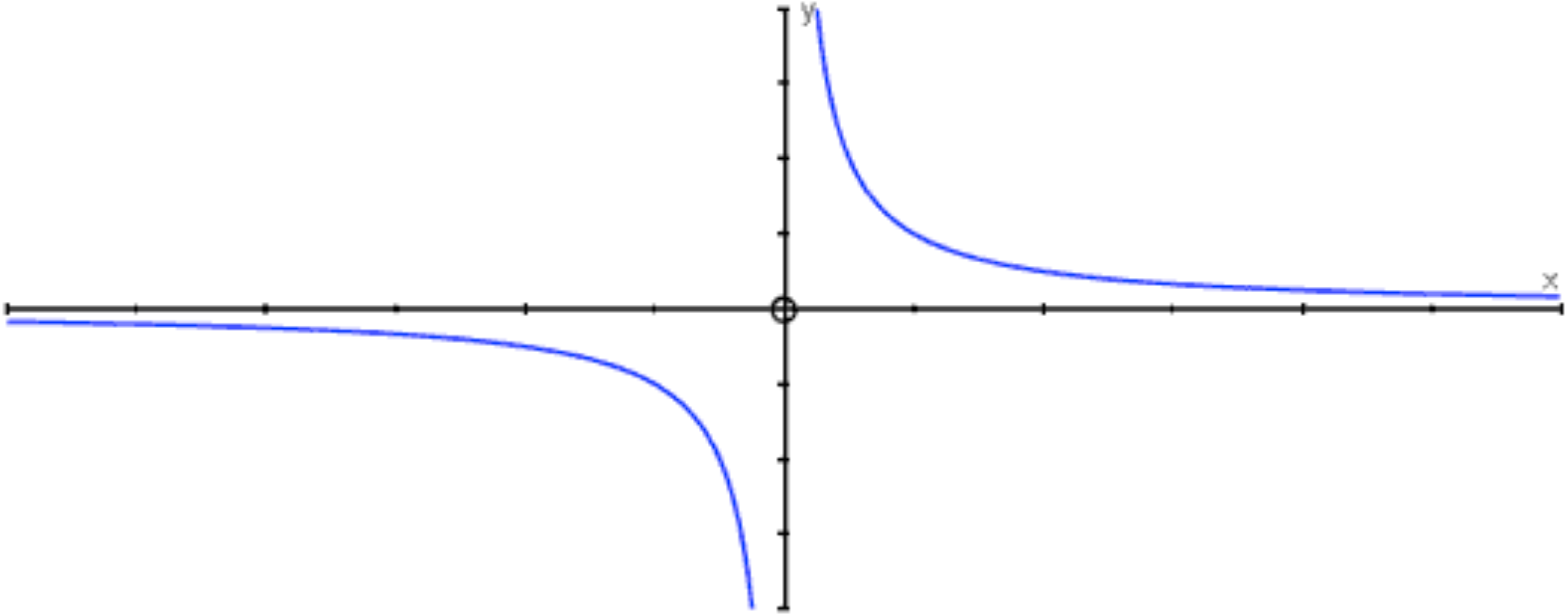
7. Positive Reciprocal

Equation: Contains a fraction with a positive x on the bottom

Examples: $y = 1/x$ $y = 5/x$ $y = 7/2x$ $y = 3/4x + 2$



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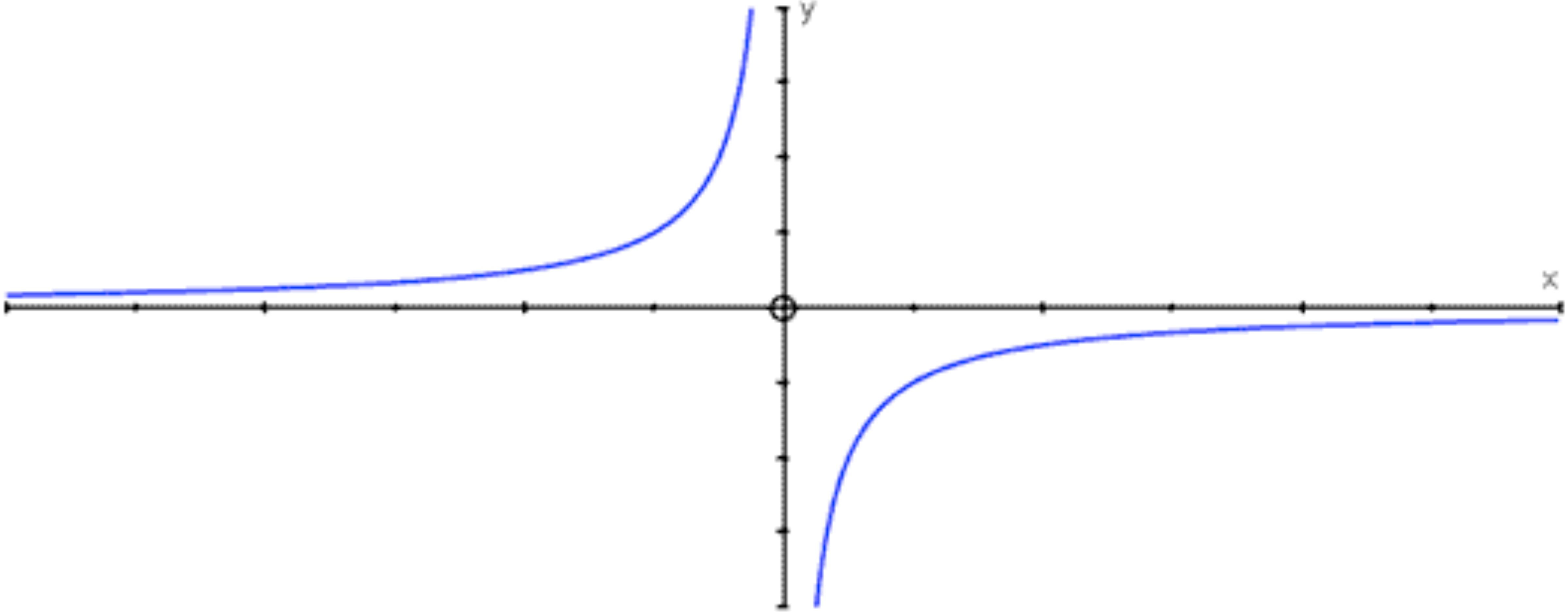
8. Negative Reciprocal

Equation: Contains a fraction with a negative x on the bottom

Examples: $y = -\frac{1}{x}$ $y = \frac{5}{-x}$ $y = -\frac{7}{2x}$ $y = 2 - \frac{2}{x}$



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4. Travel Graphs and Story Graphs

Interpreting Travel Graphs and Story Graphs

Often you will be presented with a "real life" graph and asked a few question based upon it.
Now, the temptation is to rush in and write down the first thing that you see...

But don't!

Just take a few moments, and ask yourself these questions before your pen touches the paper!

1. Look carefully at both **axis** to see what the variables are

2. Look at the **scale** carefully so you can accurately read the graph

3. Look at the **gradient** of the graph:
  - What does a horizontal line mean?
  - What does a positive/negative slope mean?

4. Always **read** the question extremely careful and **check** your answer!

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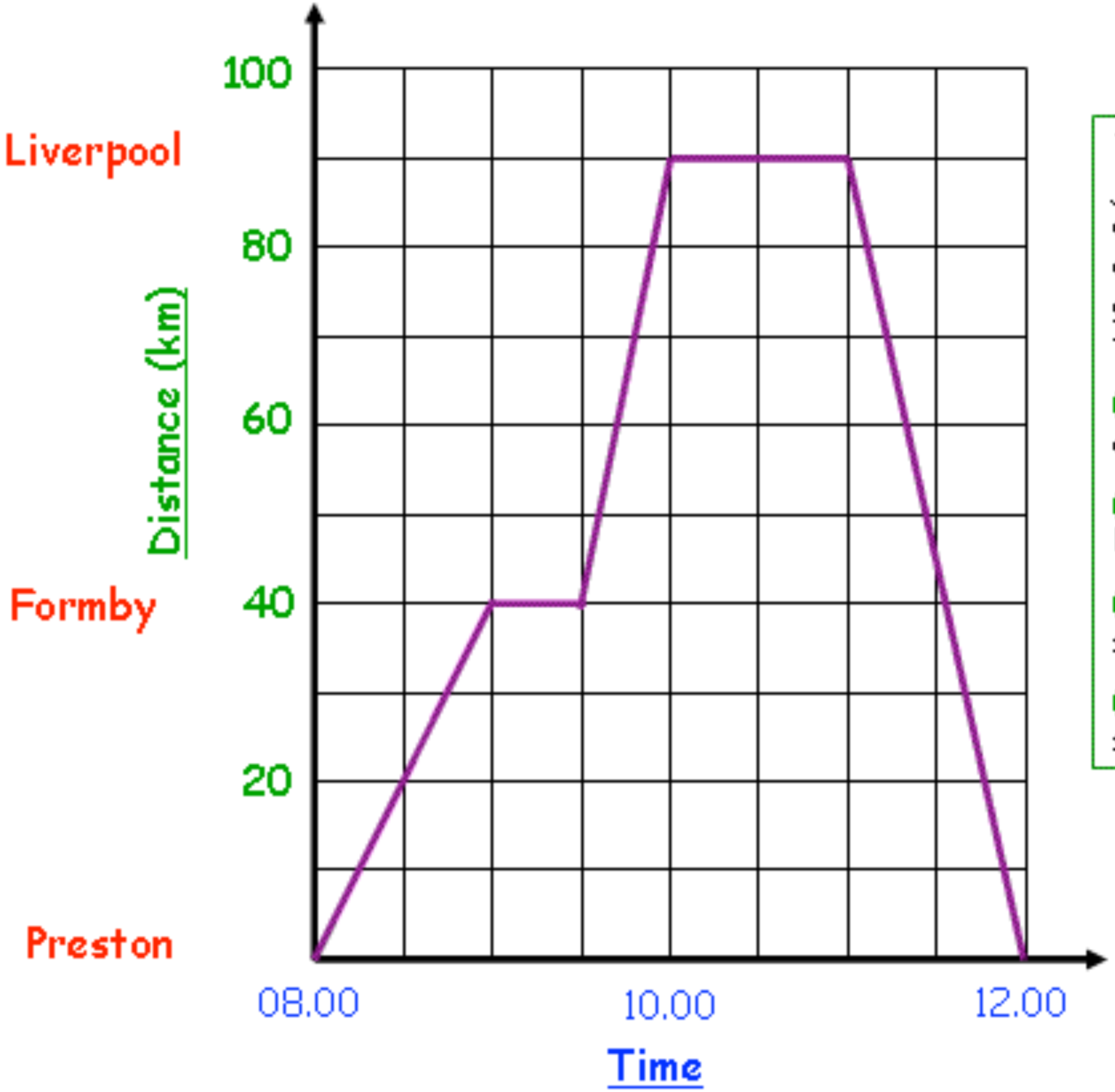
Example 1 – Travel Graph



The graph on the left shows a journey made by a family in a car between Preston, Formby and Liverpool. Look at the graph and then answer the following questions:

(a) What time did the family arrive in Liverpool?

(b) What is the distance from Formby to Liverpool?

(c) How long did the family spend not moving?

(d) What was the average speed on the journey home?

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Before we begin...

Okay, let's get to the bottom of what this graph is showing us by asking ourselves those key questions:

1. Look carefully at both **axis** to see what the variables are

Okay, so we have distance in kilometres going up the y axis, and time in hours going along the x axis

2. Look at the **scale** carefully so you can accurately read the graph

On the y axis every square represents 10km, and on the x axis every square is 30 minutes (quarter of an hour)

3. Look at the **gradient** of the graph:

- What does a horizontal line mean?

A horizontal line means that time is still passing, but the distance travelled isn't changing... so the family must have stopped moving!

- What does a positive/negative slope mean?

Positive slopes mean the family is travelling from Preston towards Liverpool, and a negative slope means they are on their way back home!

Note: If you wanted to be really clever (and why not!) you could say that the family are travelling faster between Formby and Liverpool than between Preston and Formby.
Why?... well, notice how the line is steeper, meaning they are travelling more distance in less time, so they must be going quicker!

4. Okay, now we have a really good understanding of the graph, so we can answer all the questions... and hopefully it will be dead easy!

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Answering the Questions:

(a) What time did the family arrive in Liverpool?

The line first hits Liverpool at 10.00

(b) What is the distance from Formby to Liverpool?

Formby is 40km from Preston, Liverpool is 90km from Preston, so the distance from Formby to Liverpool must be 50km!

(c) How long did the family spend not moving?

As we discussed, when the family is not moving we see a horizontal line. Well, that happens twice, firstly at Formby for 30 minutes, and then at Liverpool for 60 minutes, giving us a grand total of 90 minutes… or one and a half hours!

(d) What was the average speed on the journey home?

Okay, this is the tricky one. To answer it you need to know that:

Average Speed = Distance Travelled ÷ Time Taken

Which means on the journey home we have:

Average Speed = 90 km ÷ 1 hour
= 90 km/hr

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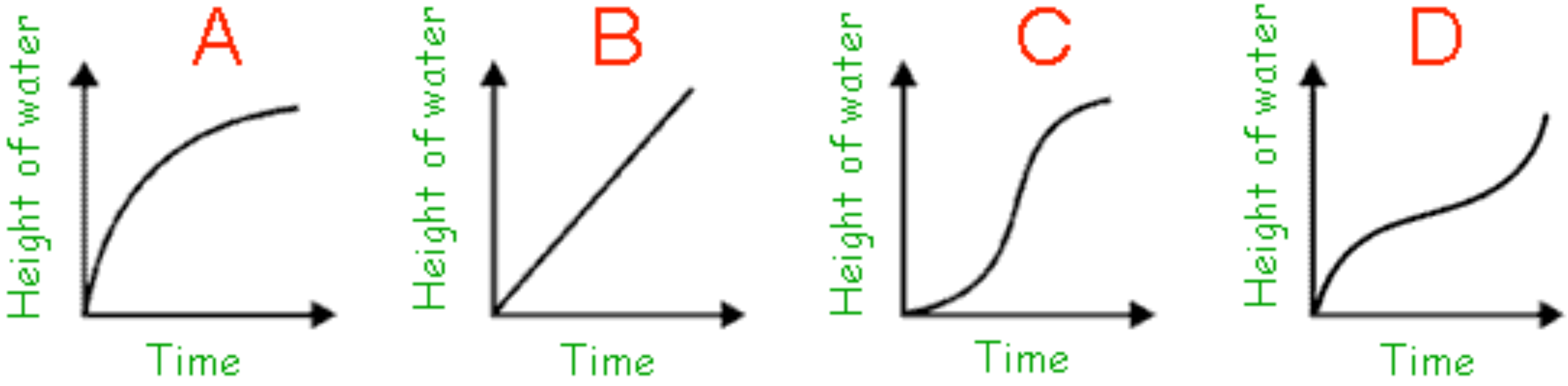
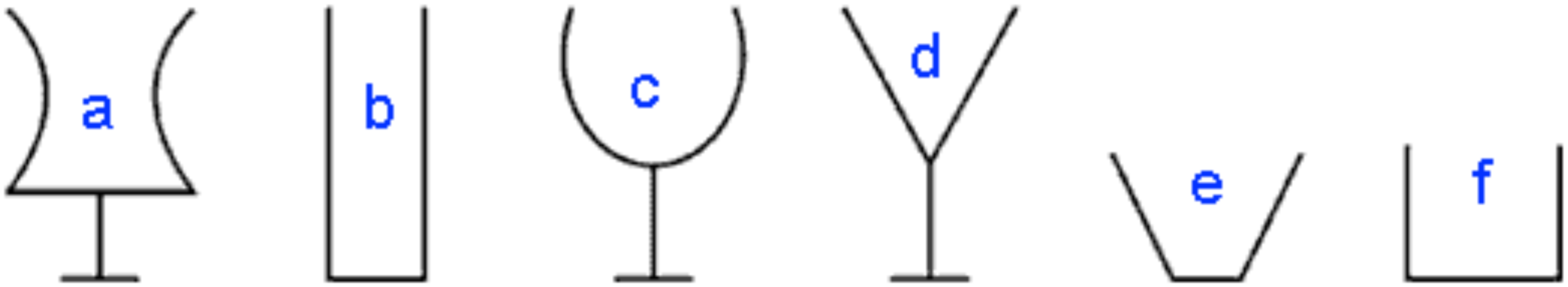
Example 2 – Story Graph

Water is poured into various glasses at a constant rate. The graphs below are sketches showing how the height of water in the glasses changes over time. Match up the shape of the glasses with their graphs

Note: Each graph can represent more than one glass.





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Before we begin…

Okay, this is a bit trickier, so once more let's get to the bottom of what these graphs are showing us by asking ourselves those key questions:

1. Look carefully at both **axis** to see what the variables are
Okay, so we have height of water going up the y axis, and time going along the x axis

2. Look at the **scale** carefully so you can accurately read the graph
There is no scale, so this doesn't matter
Note: This is also the reason why more than one glass can match to each graph!

3. Look at the **gradient** of the graph:
Okay, I am going to change the questions slightly here as this is the key to this problem:

What does a straight line mean?
The height of the water is changing by the same amount as time passes… so the sides of the glass must be straight!

What does a curved line mean?
Well, it depends on the shape of the curve, but generally a curved line means that the height of the water is not changing by the same amount, so the sides of the glass must also be curved

4. Okay, like I say, this question is a lot trickier than the first, so have a go at it and then have a look at my answers.

Try to picture that water dropping constantly into those glasses and what the height of the water will be doing!

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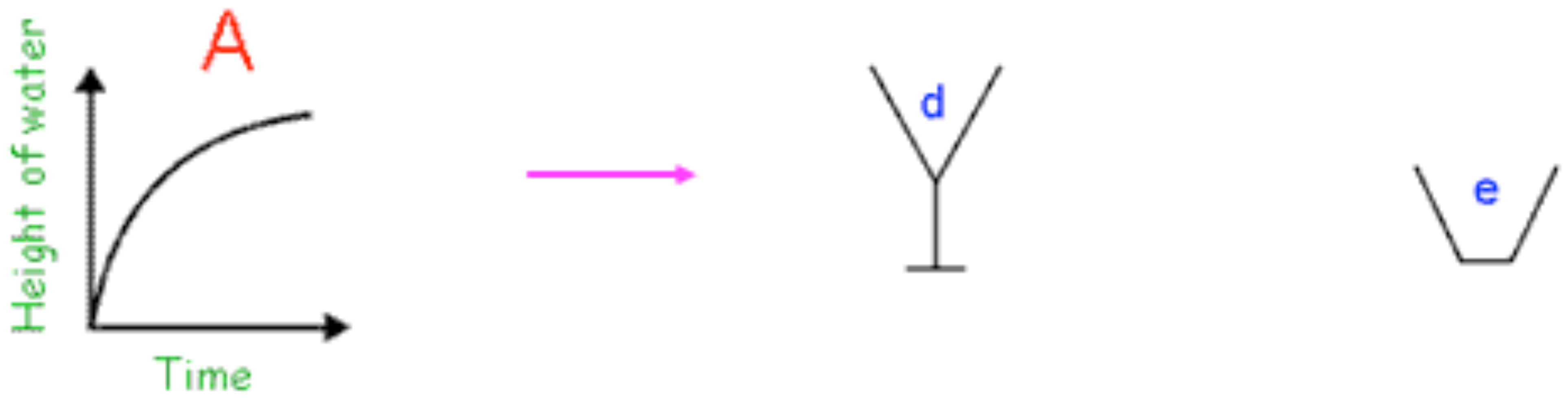
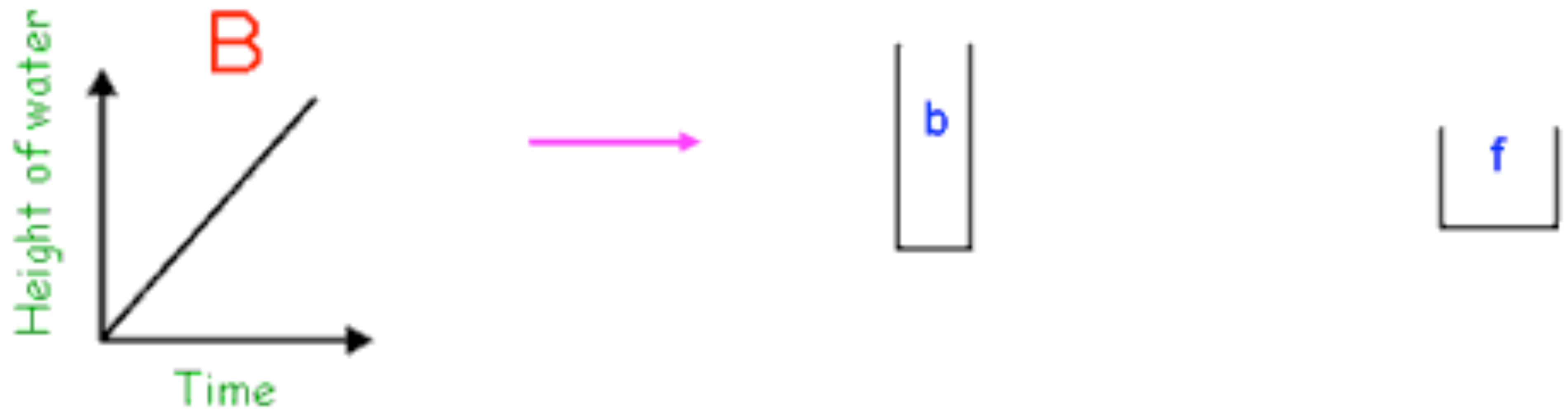
Answering the Question:





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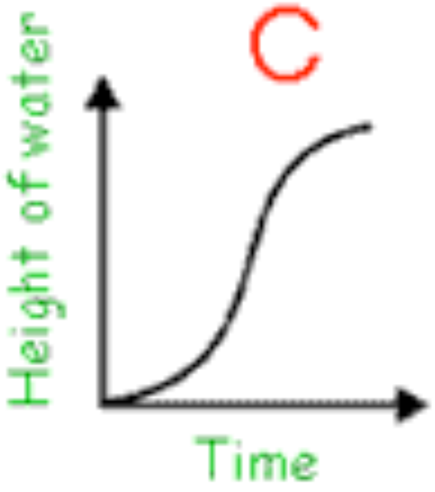

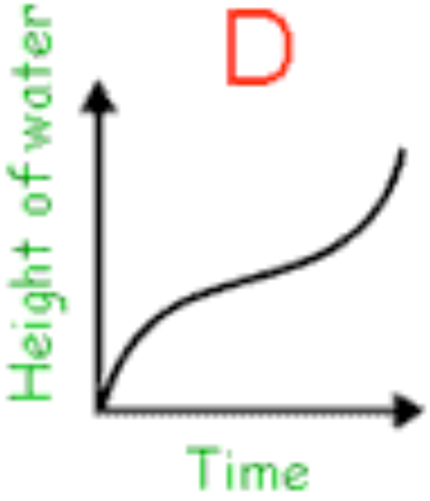

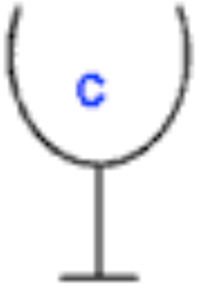

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Shape, Space and Measure

1. Angle Facts

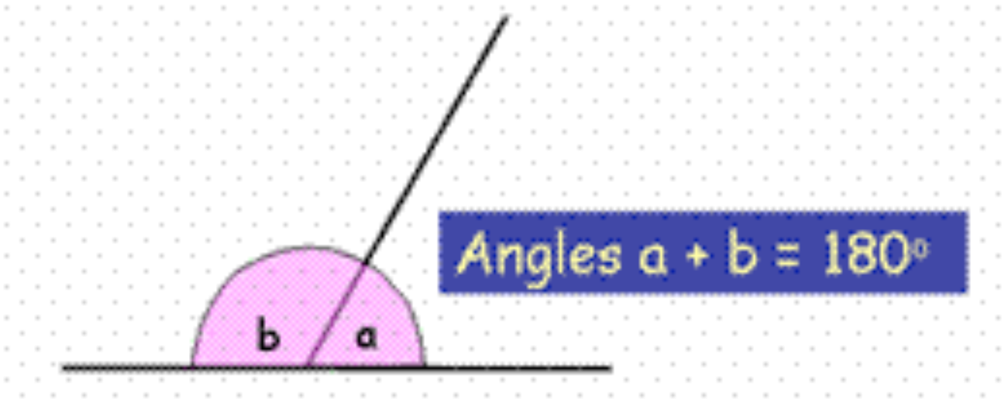
Three things you should Learn about Angle Facts:

1) What each of the facts say
2) How to spot them
3) How to show you are using angle facts in your answers

And if you can do all these, then it's goodbye to another topic off our list!

Fact 1: Angles on a Straight Line

Fact: Angles on a straight line add up to 180^0



How to spot it: Find any continuous straight line, with another straight line joining it or cutting across it

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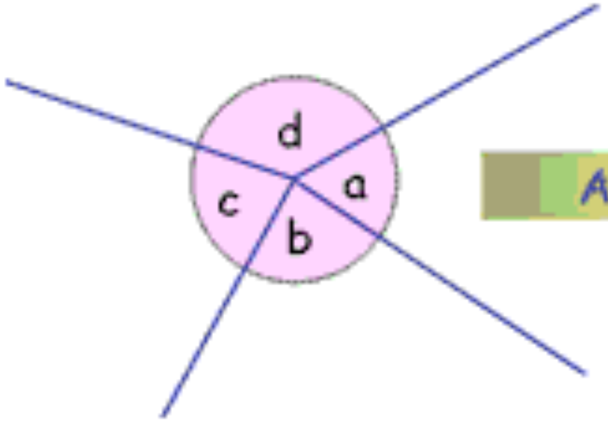
Fact 2: Angles around a Point

Fact: Angles around a point add up to 360^0

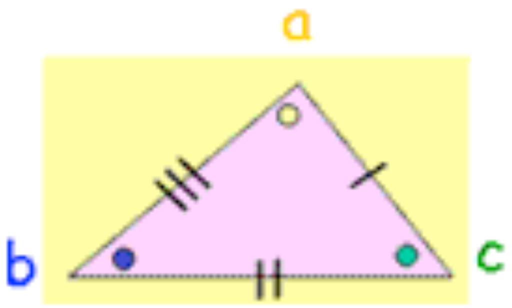


Angle $a + b + c + d = 360^0$

How to spot it: If you have a collection of lines all crossing at one point, then it's time to use this rule!

Fact 3: Angles in a Triangle

Fact: The interior (inside) angles of a triangle add up to 180^0



$a + b + c = 180^0$

How to spot it: Find any type of triangle (equilateral, isosceles, right-angled, or scalene) and all the angles inside will add up to 180^0

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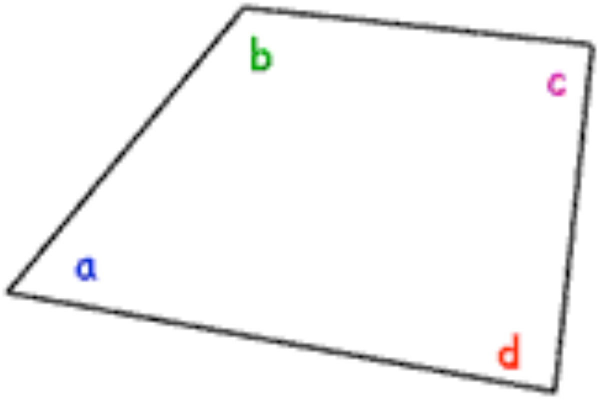
Fact 4: Angles in a Quadrilateral

Fact: Interior (inside) angles of a quadrilateral add up to 360^0

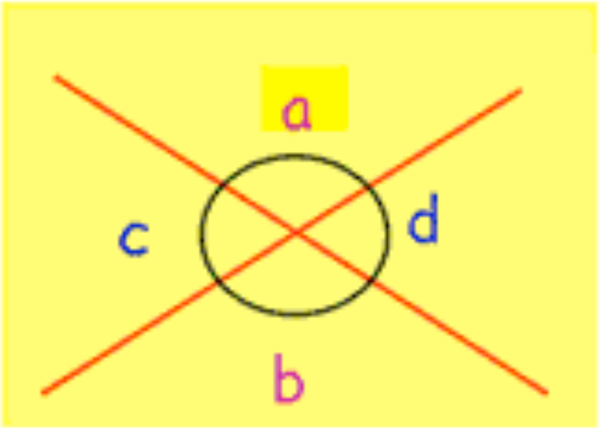


$a + b + c + d = 360^0$

How to spot it: Find any 4 sided shape (square, rectangle, trapezium, kite, etc.) and the inside angles will add up to 360^0

Fact 5: Opposite Angles

Fact: Opposite Angles are equal



$a = b$

$c = d$

How to spot it: Find two continuous straight lines crossing at a point. The pairs of angles opposite each other will be equal

Note: Using Fact 2, all the angles around that point will add up to 360^0

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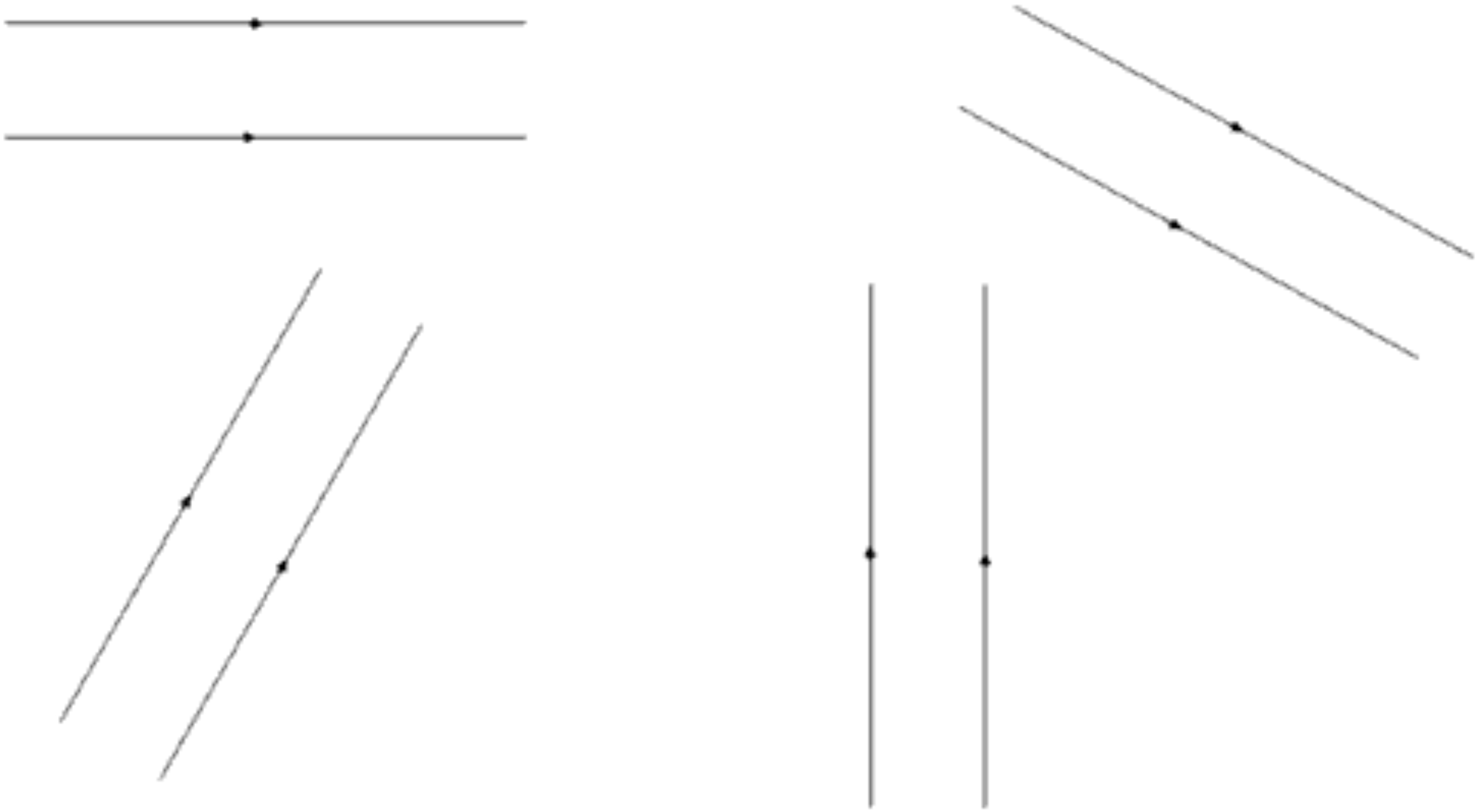
A Quick Note on Parallel Lines

For these next 3 Angle Facts, you need to be comfortable with Parallel Lines...

Parallel lines are lines which never meet, and always keep a perfectly equal distance apart.

Remember: Only assume lines are parallel if they have those little arrows on them:



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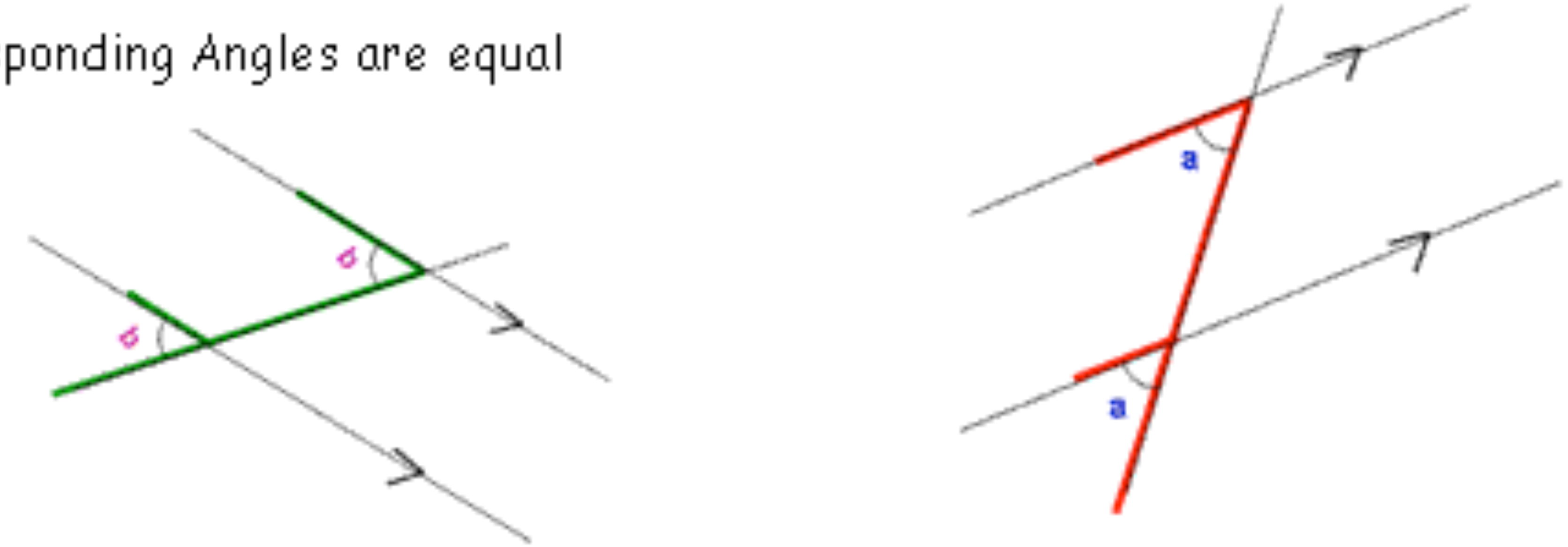
Fact 6: Corresponding Angles

Fact: Corresponding Angles are equal

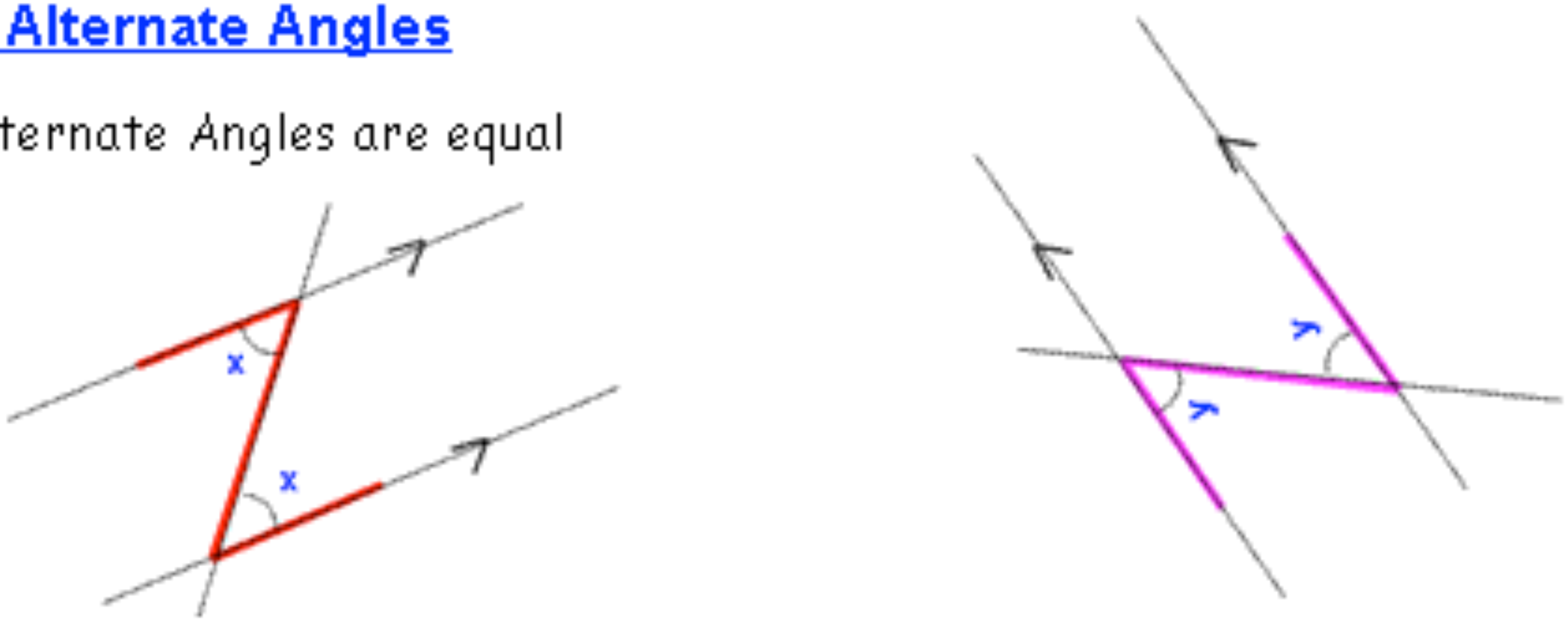


How to spot it: Look for the **F** shape, the angles underneath the arms of the F are equal

Note: The arms of the F must definitely be Parallel lines!

Fact 7: Alternate Angles

Fact: Alternate Angles are equal



How to spot it: Look for the **Z** shape, the angles "inside" the Z are equal

Note: The top and bottom of the Z must be Parallel Lines!

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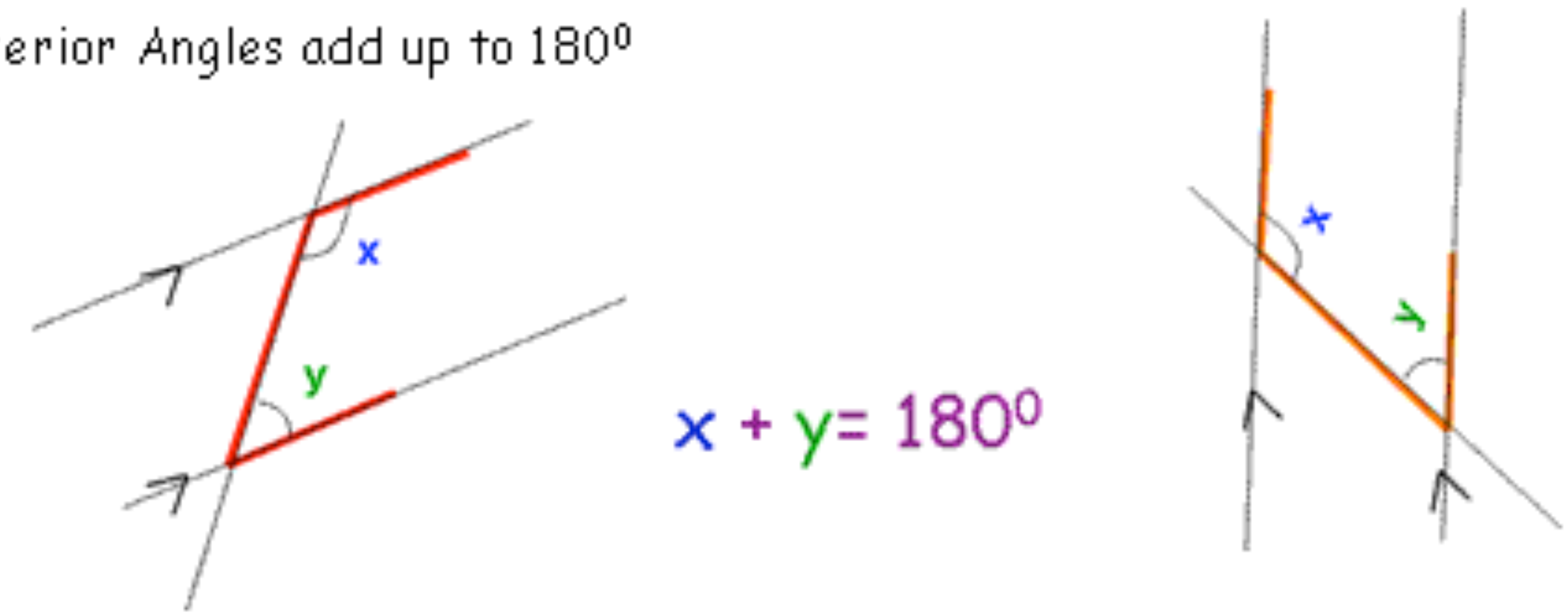
Fact 8: Interior Angles

Fact: Interior Angles add up to 180⁰



How to spot it: Look for the **C** shape, the angles underneath the top and bottom of the C add up to 180⁰

Note: The top and bottom of the C must definitely be Parallel lines!

Tips for Answering Angle Questions

1. Always write down the name of each of the Angle Facts you have used to get your answer (even if there are more than one)

2. Parallel Lines are only parallel if they have the little arrows to say so!

3. If you have lots of labelled angles to find and you just don't know where to start, sometimes it's a good idea to go in alphabetical order!

4. Often there are lots of different ways of working out the answer

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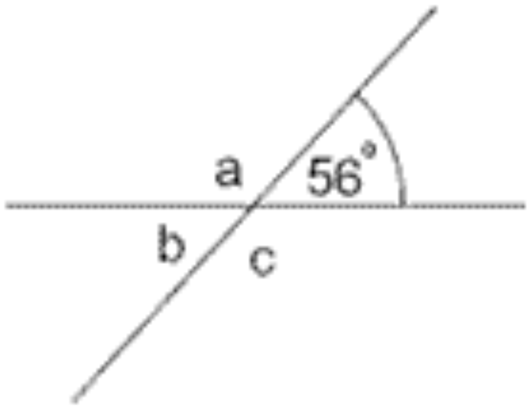
Example 1



a = 180 - 56 = 124^0

(Fact 1 – angles on a straight line)

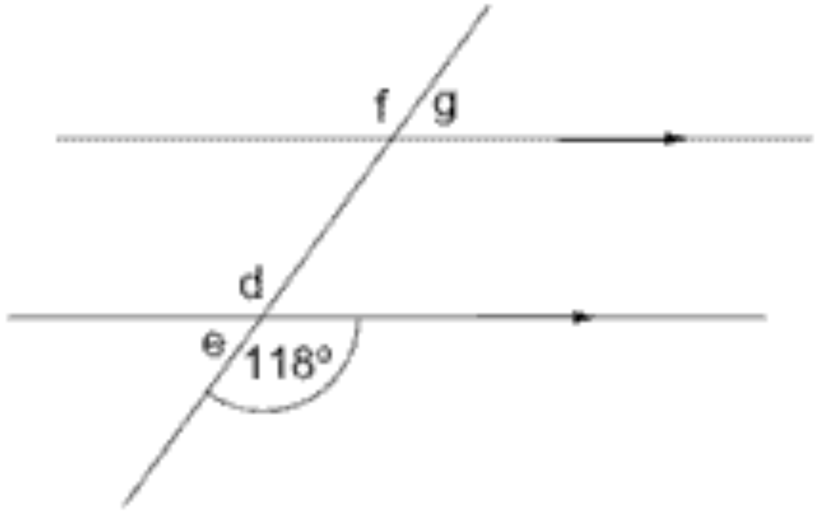
b = 56^0

(Fact 5 – opposite angles)

c = 360 - 56 - 124 - 56 = 124^0

(Fact 2 – angles around a point)

Example 2



d = 118^0

(Fact 5 – opposite angles)

e = 180 - 118 = 62^0

(Fact 1 – angles on a straight line)

f = 118^0

(Fact 6 – corresponding angles)

g = 180 - 118 = 62^0

(Fact 1 – angles on a straight line)

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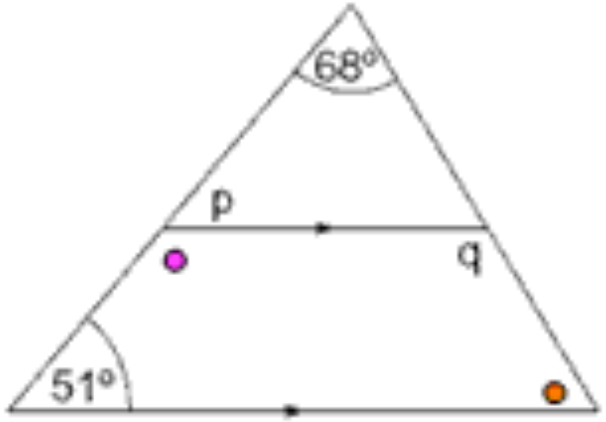
Example 3



p = 51^0

(Fact 6 – corresponding angles)

To work out q:

● = 180 - 51 = 129^0 (Fact 1 – angles on a straight line)

● = 180 - 51 - 68 = 61^0 (Fact 3 – angles in a triangle)

q = 360 – 51 – 129 – 61 = 119^0

(Fact 4 –angles in a quadrilateral)

Example 4

r = 180 - 106 - 35 = 39^0

(Fact 3 – angles in a triangle)

s = 39^0

(Fact 6 – corresponding angles)

t = 180 - 39 = 141^0

(Fact 8 – interior angles)

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2. Polygons

One of Mr Barton's Top 10 Maths Jokes

What did the pirate (who was also a very keen mathematician) say when his parrot flew away?... "Poly-gon!"... you can't beat a maths joke, hey?... anyway...

What are Polygons?

A Polygon is any closed shape which has three or more sides.

Regular Polygons

All their sides are the same length, and all their angles are the same size

e.g. squares, equilateral triangles, regular octagons...

Irregular Polygons

You've guessed it... these do not have equal length sides and angles

Rectangle, kites and trapeziums are an irregular polygons, but so too are shapes like this:

Two types of Polygons that you must be especially clued up about are quadrilaterals and triangles

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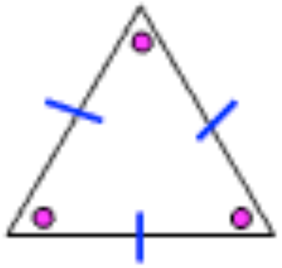
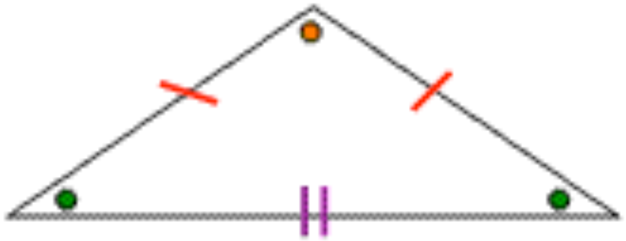
1. Triangles

There are 4 types of triangles you need to be on the look-out for and you must know the properties of (what is special about) each of them



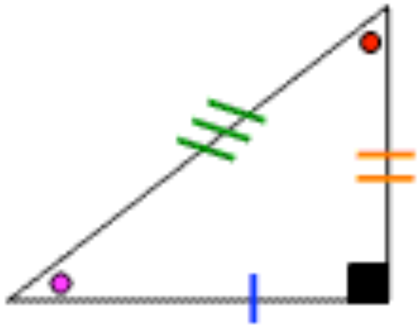
Equilateral

- All angles are equal (60° each)
- All sides are the same length
- Three lines of symmetry



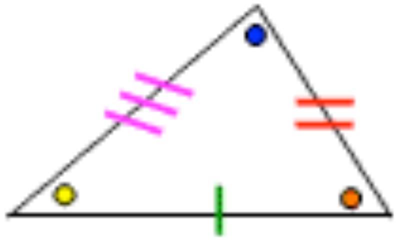
Isosceles

- Two angles are equal
- Two sides are the same length
- One line of symmetry



Right Angled

- One angle is 90°
- All sides may be different lengths
- All angles may be different
- May have 0 or 1 line of symmetry



Scalene

- All angles are different sizes
- All sides are different lengths
- No lines of symmetry

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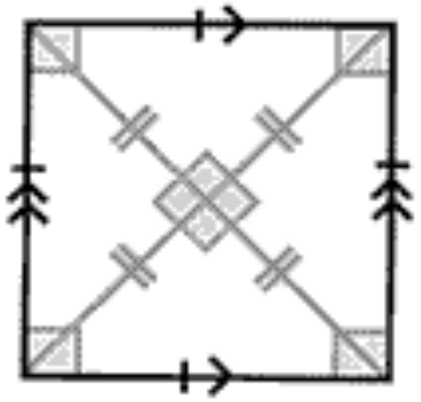
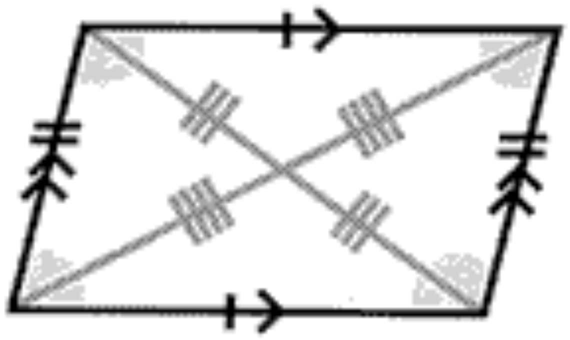
2. Quadrilaterals

A Quadrilateral is any four-sided shape. There are lots of quadrilaterals flying around, and it is important that you know the properties of each... so here they are!



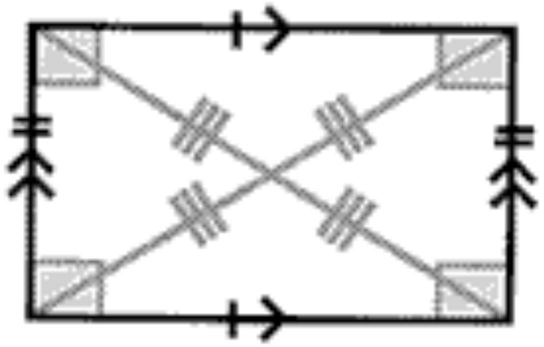
Square

- All angles are right-angles (90^0 each)
- All sides are the same length
- Two pairs of parallel lines
- Four lines of symmetry



Parallelogram

- Opposite angles are equal
- Opposite sides are the same length
- Two pairs of parallel sides
- May have no lines of symmetry



Rectangle

- All angles are right-angles (90^0 each)
- Opposite sides are the same length
- Opposite sides are parallel
- Has two lines of symmetry

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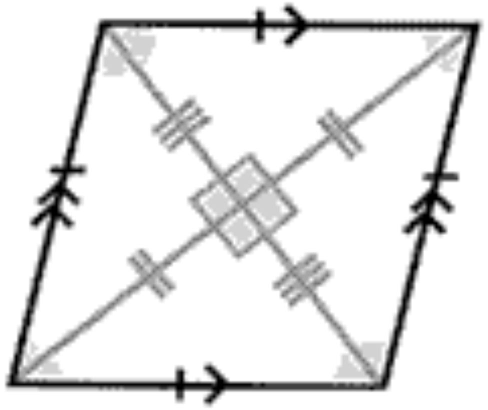
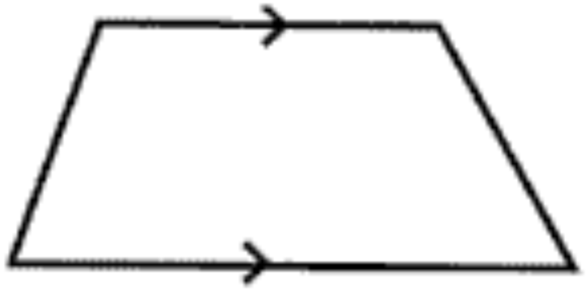


Rhombus

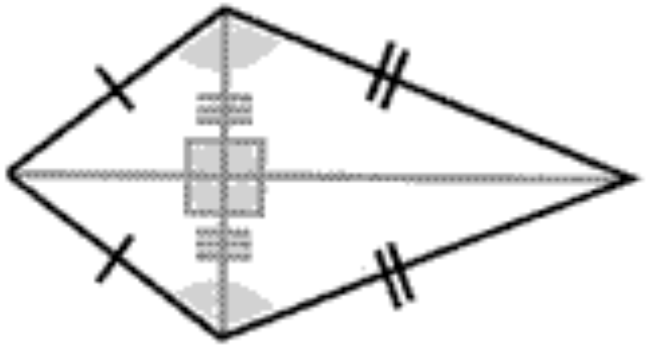
- Opposite angles are equal
- All sides are the same length
- Opposite sides are parallel
- Two lines of symmetry

Notice: Each of the four shapes above are very similar… in fact, they are all just special types of parallelograms! See how they each have two pairs of parallel sides… and then it just certain other properties that make them different shapes!



Trapezium

- All angles may be different sizes
- All sides may be different lengths
- Opposite sides are parallel
- May have no lines of symmetry



Kite

- One pair of equal angles
- Adjacent sides are the same length
- No pairs of parallel sides
- One line of symmetry

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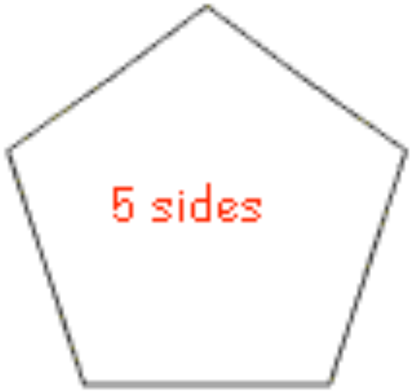
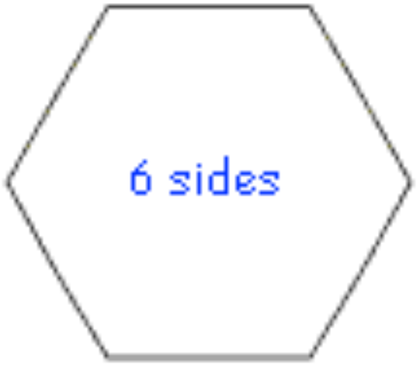
3. Other Polygons

As soon as you get above 4 sides, the names of the polygons start to get a bit weird. Here are some of the main ones you should learn.

Notice: Each of the shapes below are regular polygons as all the sides and angles are the same… but any 8 sided shape is still an octagon, it may just be an irregular one!

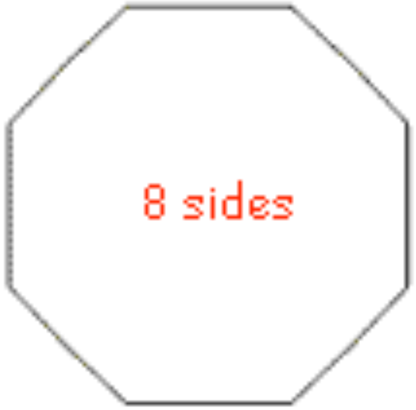


Pentagon

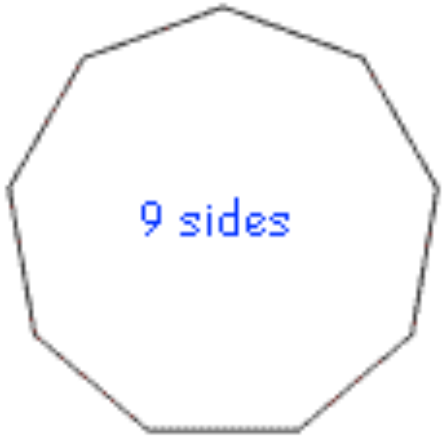


Hexagon

Heptagon / Heptagon



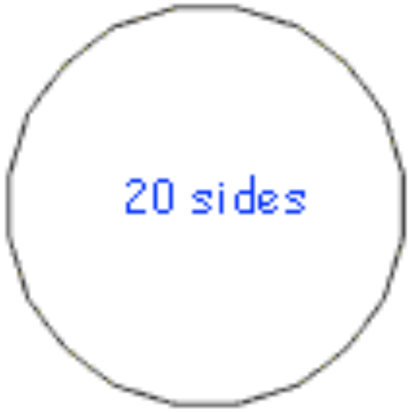
Octagon



Nonagon

Decagon

Dodecagon



Icosagon

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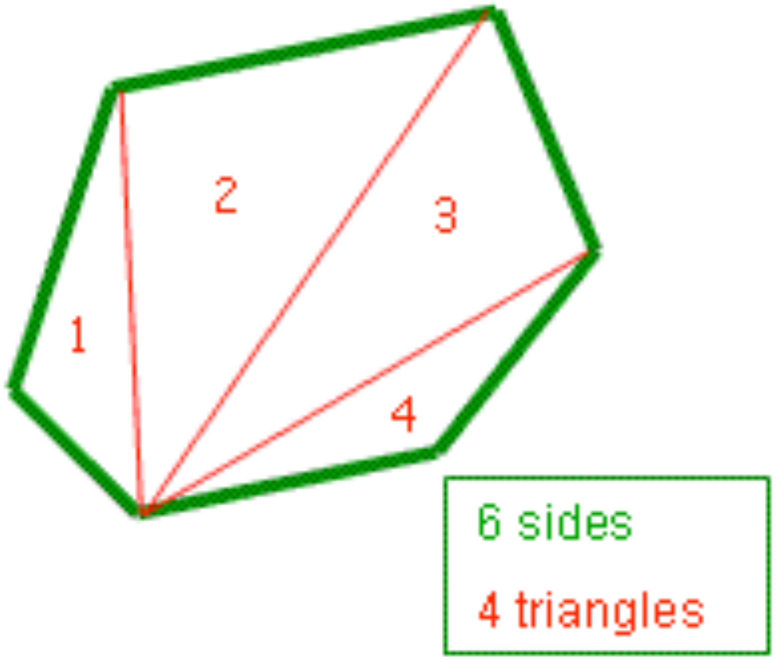
4. Interior Angles of Polygons

An interior angle is any angle inside the polygon

If we are told the number of sides a polygon has, we can work out the total sum of all the interior angles using this little formula:

Sum of all interior angles = (Number of sides of polygon – 2) x 180

Why?



Well, it's all to do with triangles…

We know that the sum of the interior angles of any triangle is 180^0, right?

Well… we can split any polygon up into triangles, like this…

And there will always be 2 fewer triangles than there are sides!

For Regular Polygons

Because all angles are equal in regular polygons, you can work out the size of each interior angle like this:

Size of each interior angle = Sum of all interior angles ÷ Number of sides

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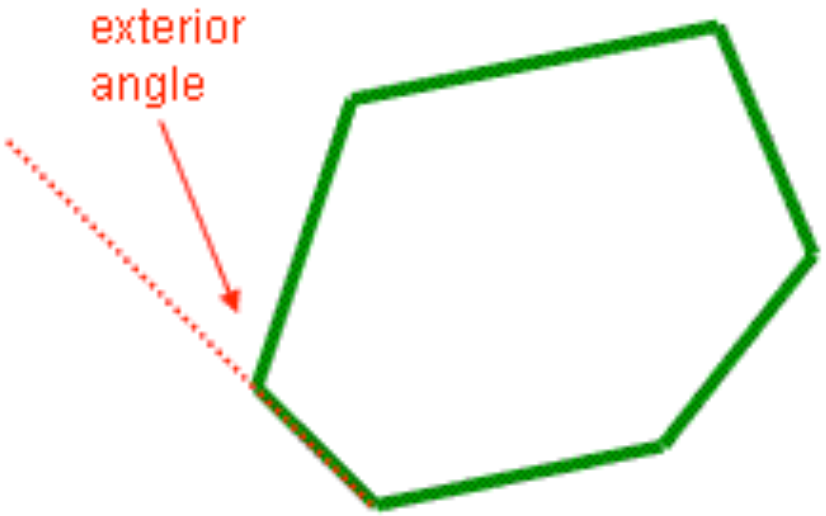
5. Exterior Angles of Polygons

An exterior angle is an angle outside the polygon made by extending one of the sides...



And here is the fact!

Sum of all exterior angles = 360^0

Why?

Well, if you keep moving around the polygon, extending the sides and measuring each exterior angle, by the time you get back to where you started you have made... a circle! Which, as we all know, contains 360^0

For Regular Polygons

If all interior angles are equal for regular polygons, then all exterior angles are equal too, so to work out the size of each one, we do this...

Size of each exterior angle = $360^0 \div$ Number of sides

Note: If you know the sizes of the exterior angles of a regular polygon, then you can also work out the sizes of the interiors by remembering that angles on a straight line add up to 180^0

Size of each interior angle = $180^0 -$ Size of each exterior angle

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6. Massive Table of Facts

Using the formulae we have talked about, it is possible to work out pretty much any angle fact about any size polygon. Have a practice to make sure you can get the numbers in this table...

Name of Polygon	Number of Sides	Total Sum of Interior Angles	Size of each Interior Angle if Regular	Total Sum of Exterior Angles	Size of each Exterior Angle if Regular
Triangle	3	180	60	360	120
Quadrilateral	4	360	90	360	90
Pentagon	5	540	108	360	72
Hexagon	6	720	120	360	60
Heptagon	7	900	128.6 (1dp)	360	51.4 (1dp)
Octagon	8	1080	135	360	45

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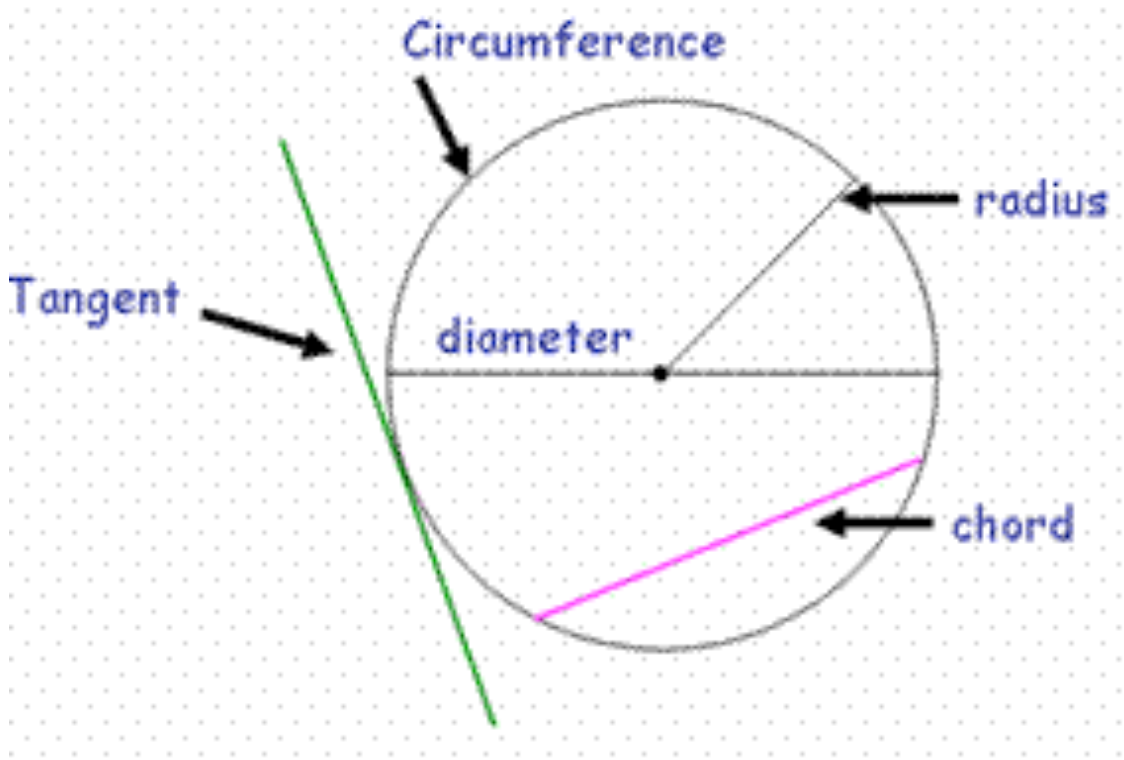
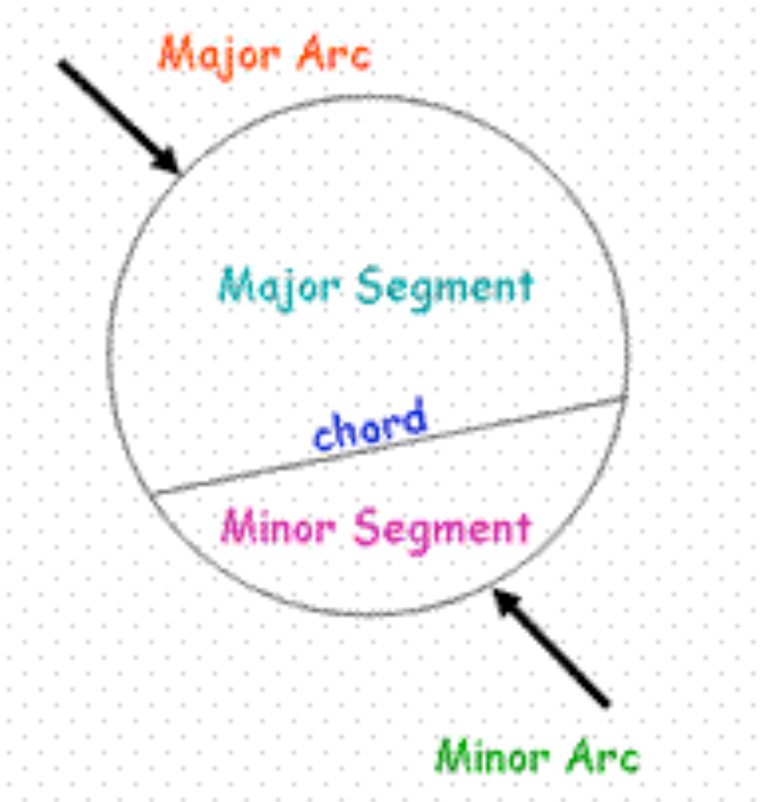
3. Circle Theorems

Parts of a Circle…

Before we start going through each of the circle theorems, it is important we know the names for each part of the circle, as we will be using these terms in this section.





Three things you should Learn about Circle Theorems:

1) What each of the theorems say
2) How to spot them
3) How to show you are using circle theorems in your answers

And if you can do all these, then that's a pretty tricky topic all sorted!

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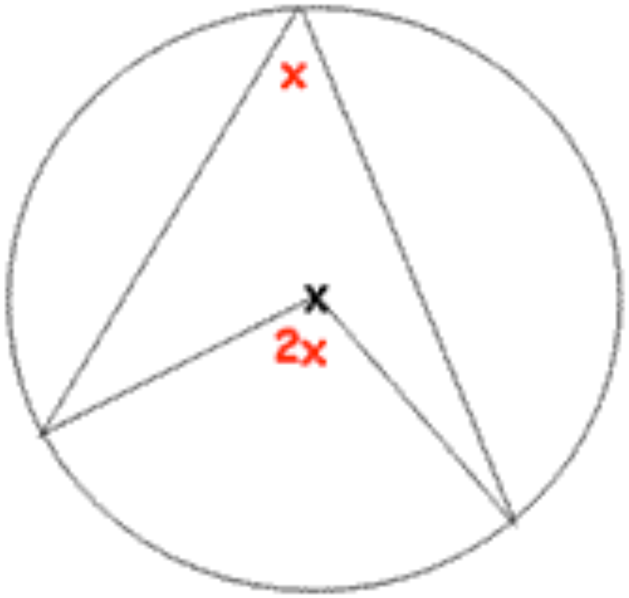
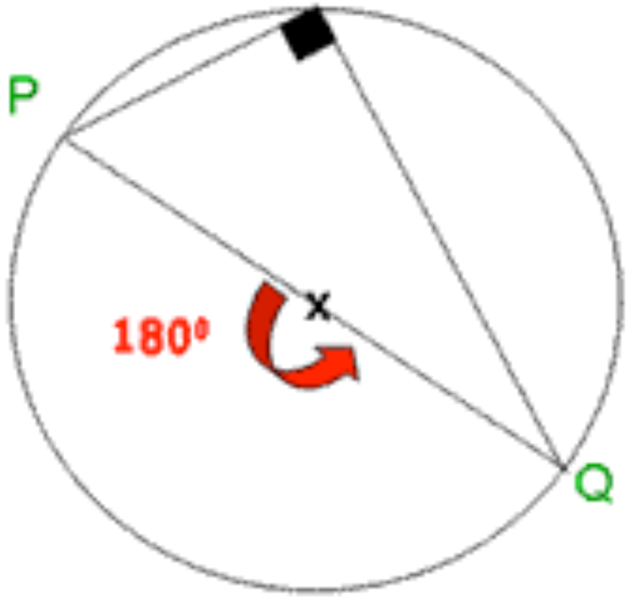
Theorem 1: Angle at the Centre



Fact: The angle at the centre is twice as big as the angle at the circumference made by the same arc or chord

How to spot it: Start with two points (could be the ends of a chord). If you go point-centre-point, the angle you make will be twice as big as if you go point-circumference-point

Theorem 2: Angles in a Semi-Circle



Fact: The angle made at the circumference in a semi circle is a right angle (90^0)

How to spot it: Look for a triangle whose base is the diameter of the circle (a line going through the centre). The angle at the circumference in this triangle will always be a right angle

Note: This theorem is just a special case of Theorem 1, because the angle at the centre when you have a straight line is 180^0, so the angle at the circumference must be half of this!

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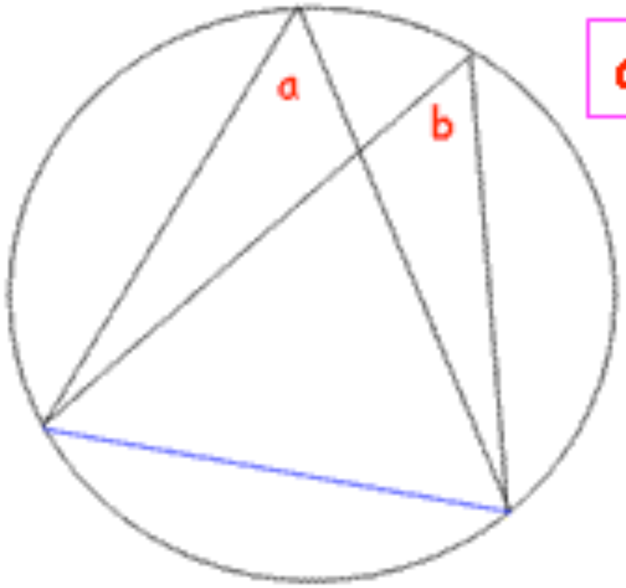
Theorem 3: Angles in the Same Segment

Fact: Angles in the same segment of a circle are equal to each other

How to spot it: Start with two points (could be the ends of a chord). If you go point-circumference-point, the angle you make will be exactly the same as if you go point-circumference-point… so long as you stay in the same segment of the circle!



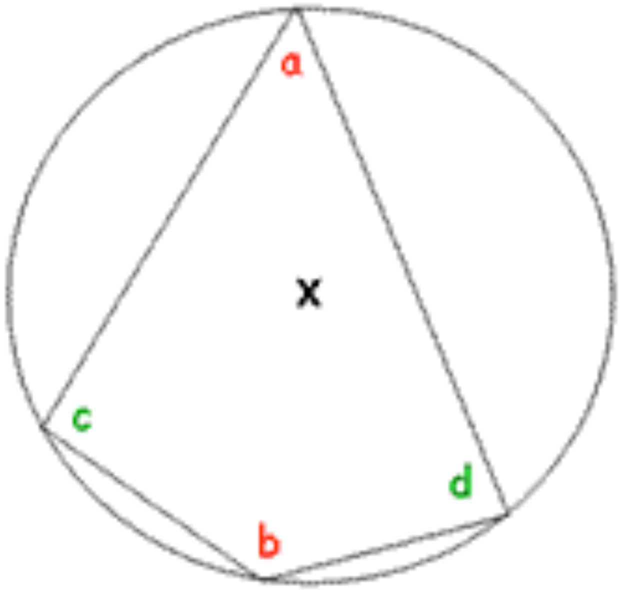
$a = b$

Theorem 4: Cyclic Quadrilateral

Fact: The opposite angles in a cyclic quadrilateral add up to 180^0

How to spot it: Look for a four-sided shape with each of the corners on the circumference. The opposite angles in this shape will always add up to 180^0

Note: Just like any other quadrilateral, the sum of all the interior angles is still 360^0



$a + b = 180^0$

$c + d = 180^0$

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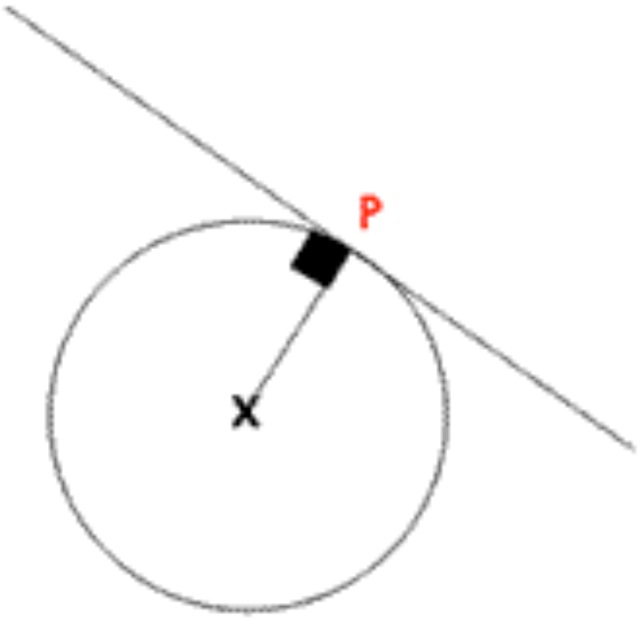
Theorem 5: Tangent

Fact: The angle made by a tangent and the radius is a right-angle (90⁰)

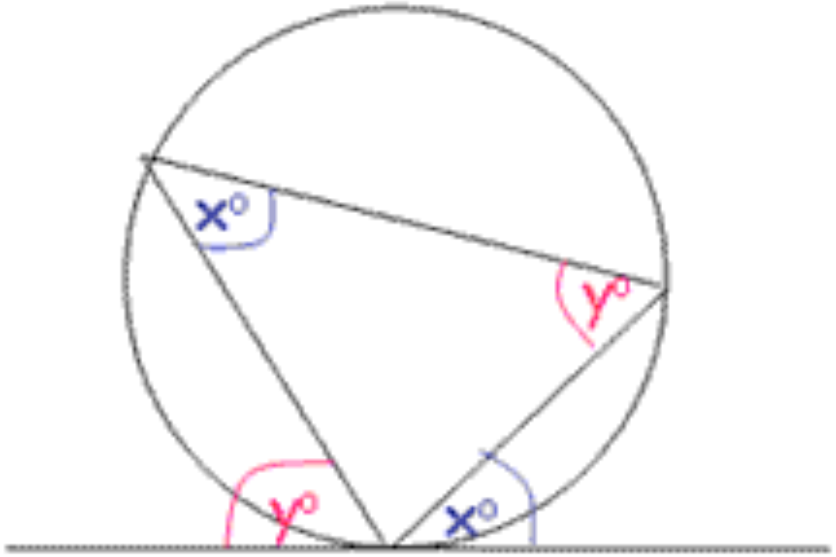
How to spot it: A tangent is a straight line that only touches a circle in one place. If you draw a line from that one place to the centre of a circle, then the angle you form is always a right-angle!



Theorem 6: Alternate Segment Theorem

Fact: The angle between a tangent and a chord at the point of contact is equal to the angle made by that chord in the other segment of the circle.

How to spot it: Look for a tangent and a chord meeting at the same point. The angle they make is exactly the same as the angle at the circumference made by that chord – imagine the chord is the base of a triangle, and the angle you want is at the top of the triangle!



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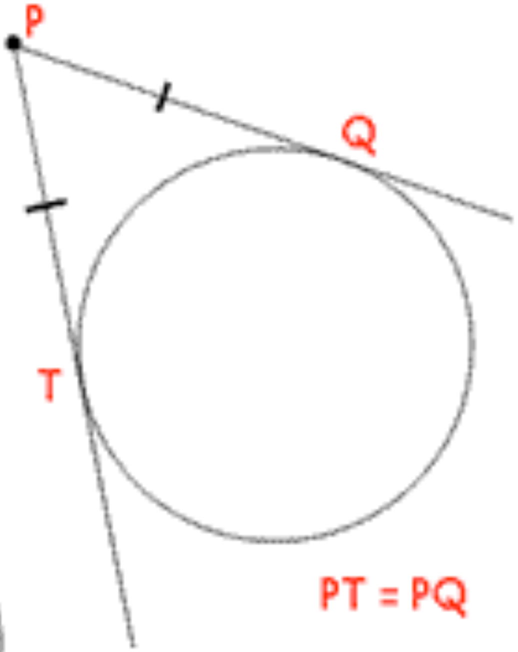
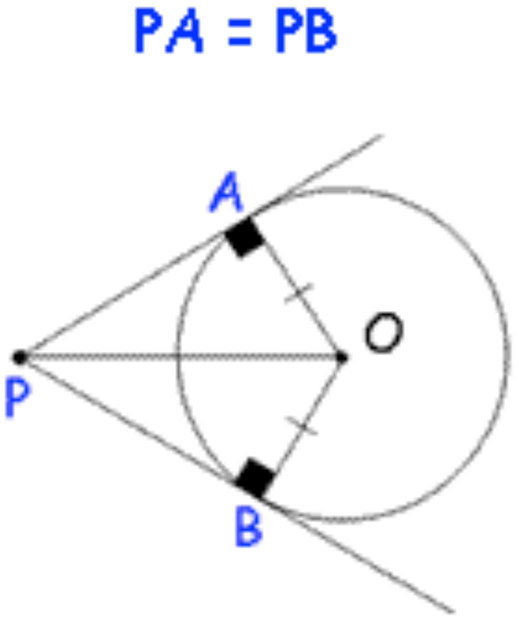
Theorem 7: Two Tangents

Fact: From any point outside the circle, you can only draw two tangents to the circle, and these tangents will be equal in length.

How to spot it: Look for where the tangents to a circle meet. The lengths between where they touch the circle and the point at which they meet will always be the same

Note: More often than not, this theorem leads to some isosceles triangles, so be on the look out!





Tips for Answering Circle Questions

1. Always write down the name of each of the Circle Theorems you have used to get your answer (even if there are more than one)

2. An angle is not a right-angle just because it looks like one! You must be able to prove it using a circle theorem, or be told it in the question!

3. To be good at circle theorems, you also need to be good at your Angle Facts – for a refresher, see 1. Angle Facts before carrying on!

4. Often there are lots of different ways of working out the answer

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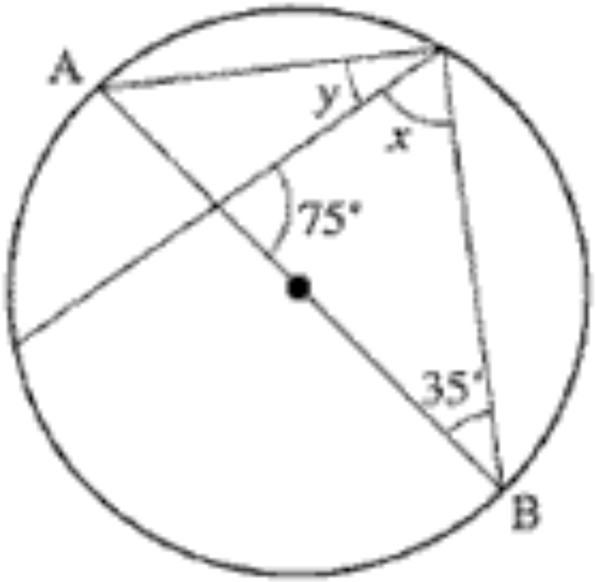
Example 1



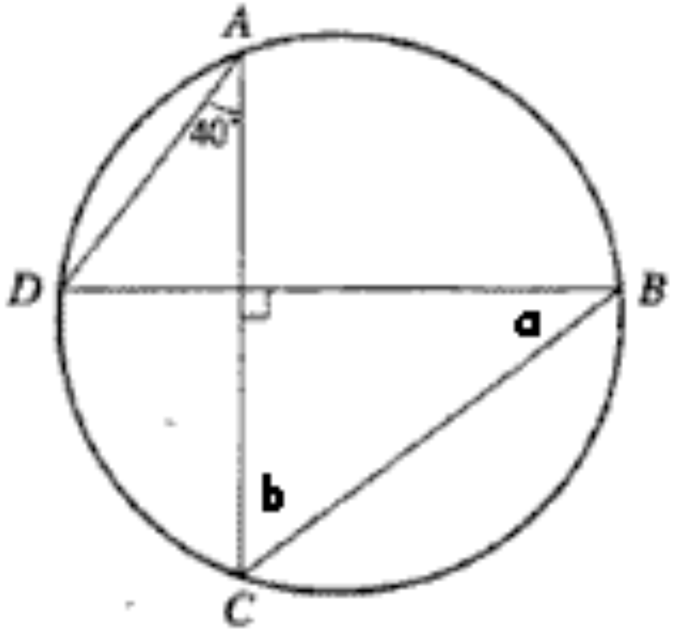
x = 180 - 75 - 35 = 70^0

(angles in a triangle)

y = 90 - 70 = 20^0

(Theorem 2 - angles in a semi-circle)

Example 2



a = 40^0

(Theorem 3 - angles in the same segment)

b = 180 - 90 - 40 = 50^0

(angles in a triangle)

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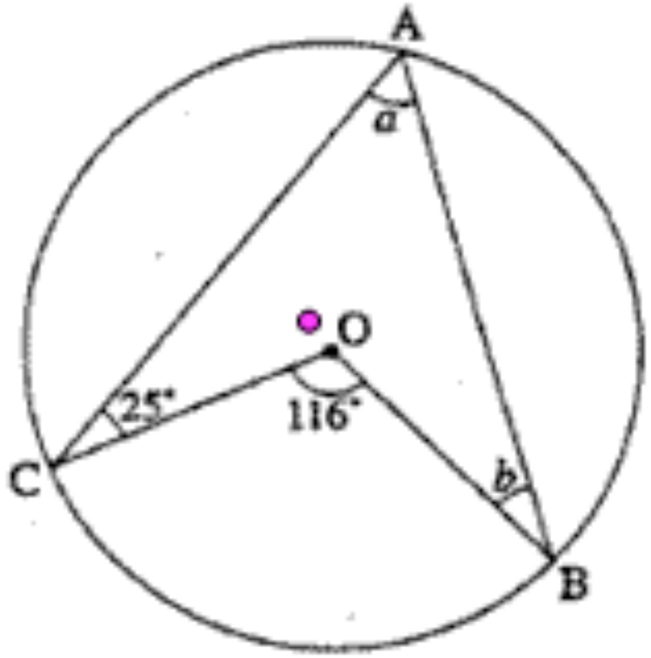
Example 3



a = 88^0

(Theorem 1 – angle at the centre)

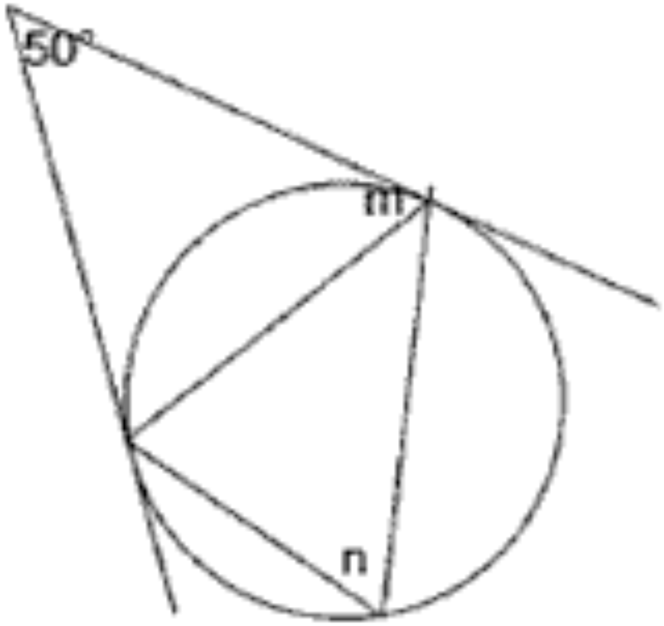
To work out b:

● = 360 - 116 = 244^0 (angles around a point)

b = 360 - 244 - 25 - 88 = 3^0 (angles in a quadrilateral)

Note: Lots of people would just put 25^0 because it looks like it… but that would be a load of rubbish!

Example 4



m = (180 – 50) ÷ 2 = 65^0

(Theorem 7 – two tangents, isosceles triangle)

n = 65^0

(Theorem 6 – alternate segment)

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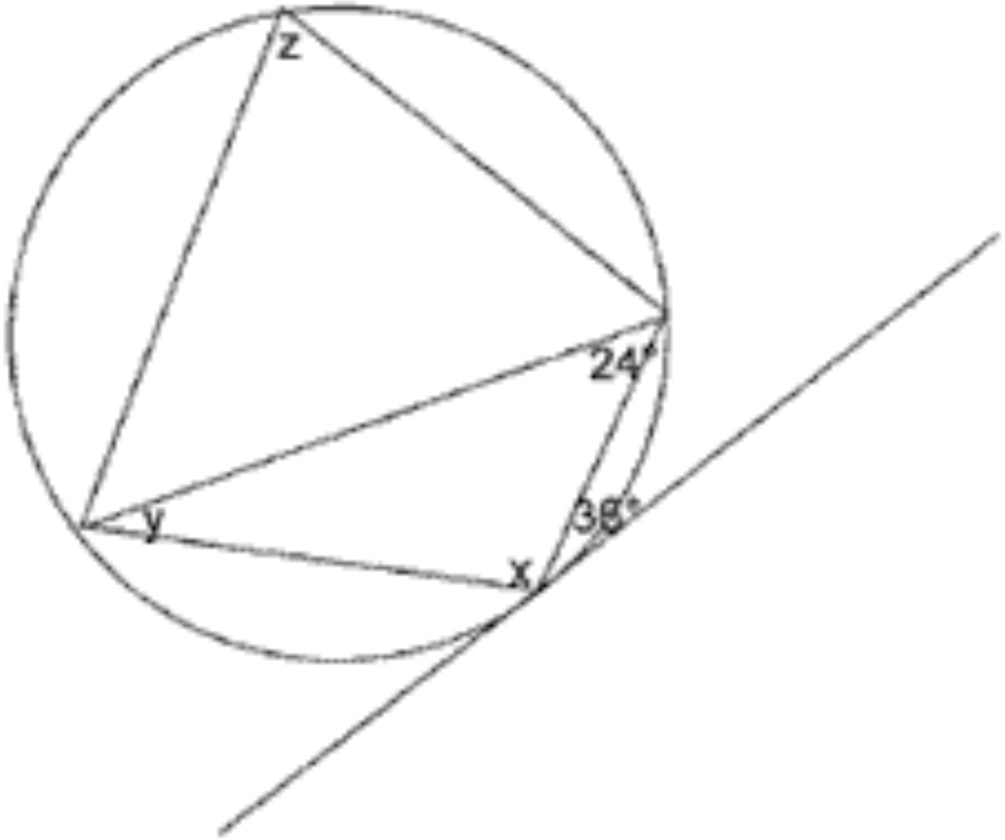
Example 5



y = 36⁰

(Theorem 6 – alternate segment)

x = 180 – 36 – 24 = 120⁰

(angles in a triangle)

z = 180 – 120 = 60⁰

(Theorem 4 – cyclic quadrilateral)

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4. Loci

What on earth is Loci?

- Loci is all about tracing the paths of points as they move following certain rules
- It has many real-life applications, especially for architects and builders who want to make sure things go in the right place and they don't run out of room
- Note: If you are one of those people who doesn't like the number and algebra bits of maths, then this could be the very topic for you!

What we are going to do in this section

- Instead of going through how to do things like draw angle-bisectors, I am going to pick out a few of the classic type of Loci questions I have seen come up in exams in the past and take you through, step-by-step, how to do each one.

NOTE: It is probably worth while reading through 8. Constructions before carrying on, as some of the skills you need are explained in greater detail there!

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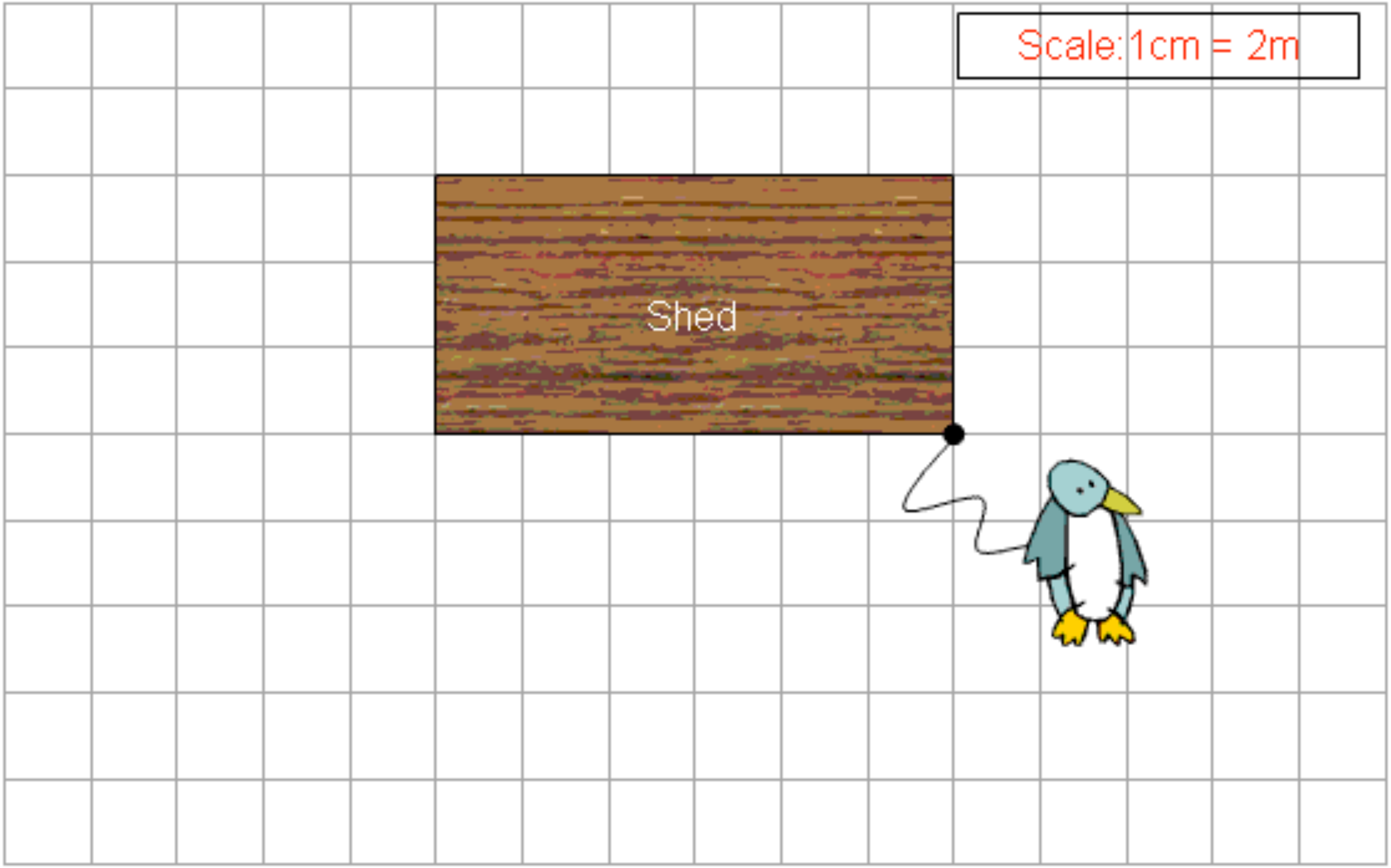
Example 1

My pet penguin has been tied up by a 10 metre rope to the corner of the shed as shown below. Draw and shade the area which my penguin can move

Skills needed: drawing circles with compass



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Steps:

1. Firstly, we need to sort out our scale – every 1cm square is equal to 2metres in real life – so the 10m rope our penguin is tied to is in fact… 5cms long!

2. Now, we want to see how far our penguin can go in all directions. So, we must draw a circle with our compass (radius 5cm) and with the centre at the point on the shed where the penguin is tied.

Watch Out! But that's not the full story… because walls of the shed prevent the penguin from going quite as far upwards – he cannot walk through walls!

He can go along the side of the shed to point B, which is 3cms away, and once he has reached this point, he can go another 2cms in any direction.

3. So… we must now set our compass again and draw a circle with radius 2cm and centre at point B.

4. We now have the area where the penguin can walk, so we can shade it in!

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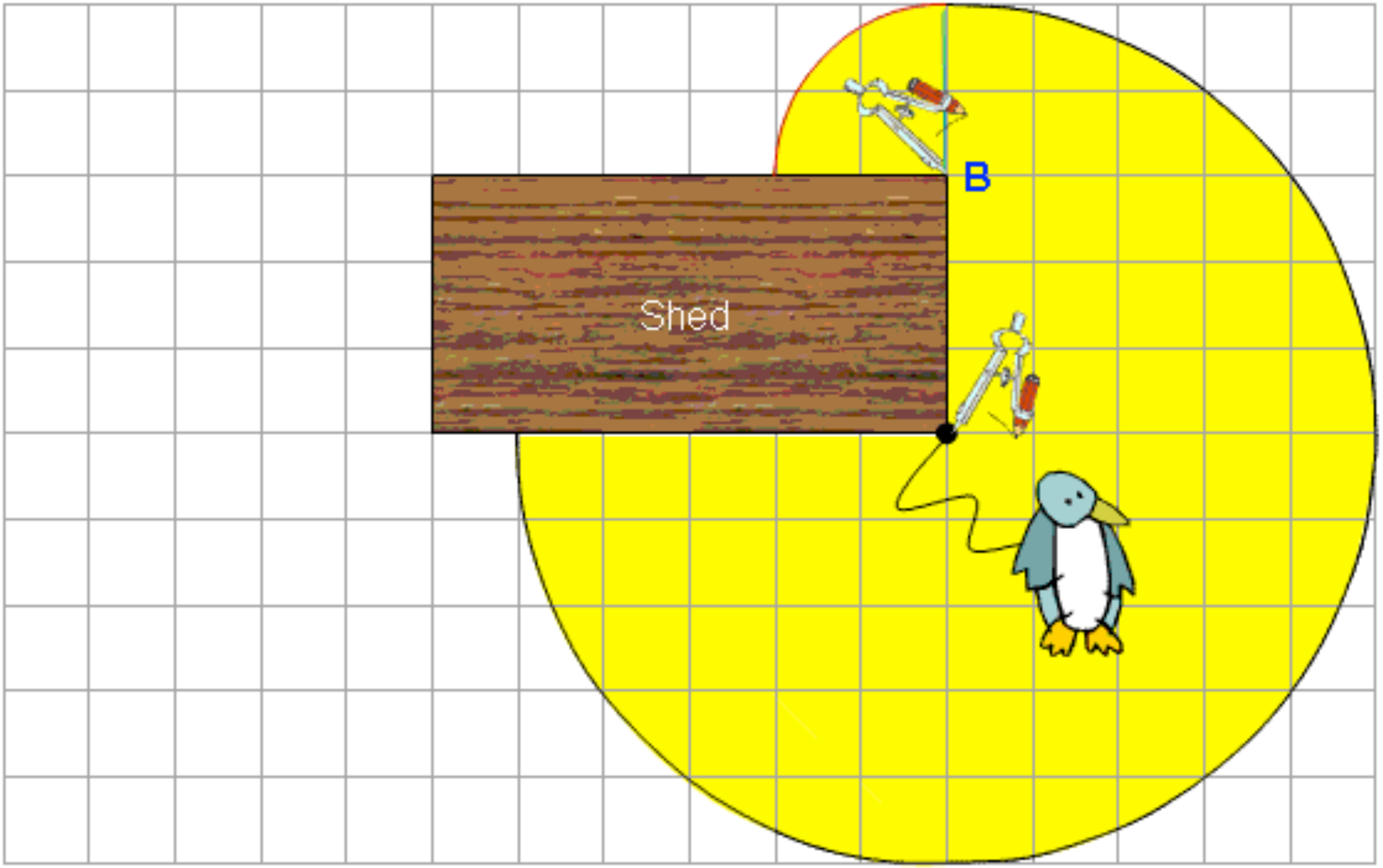
Scale:1cm = 2m



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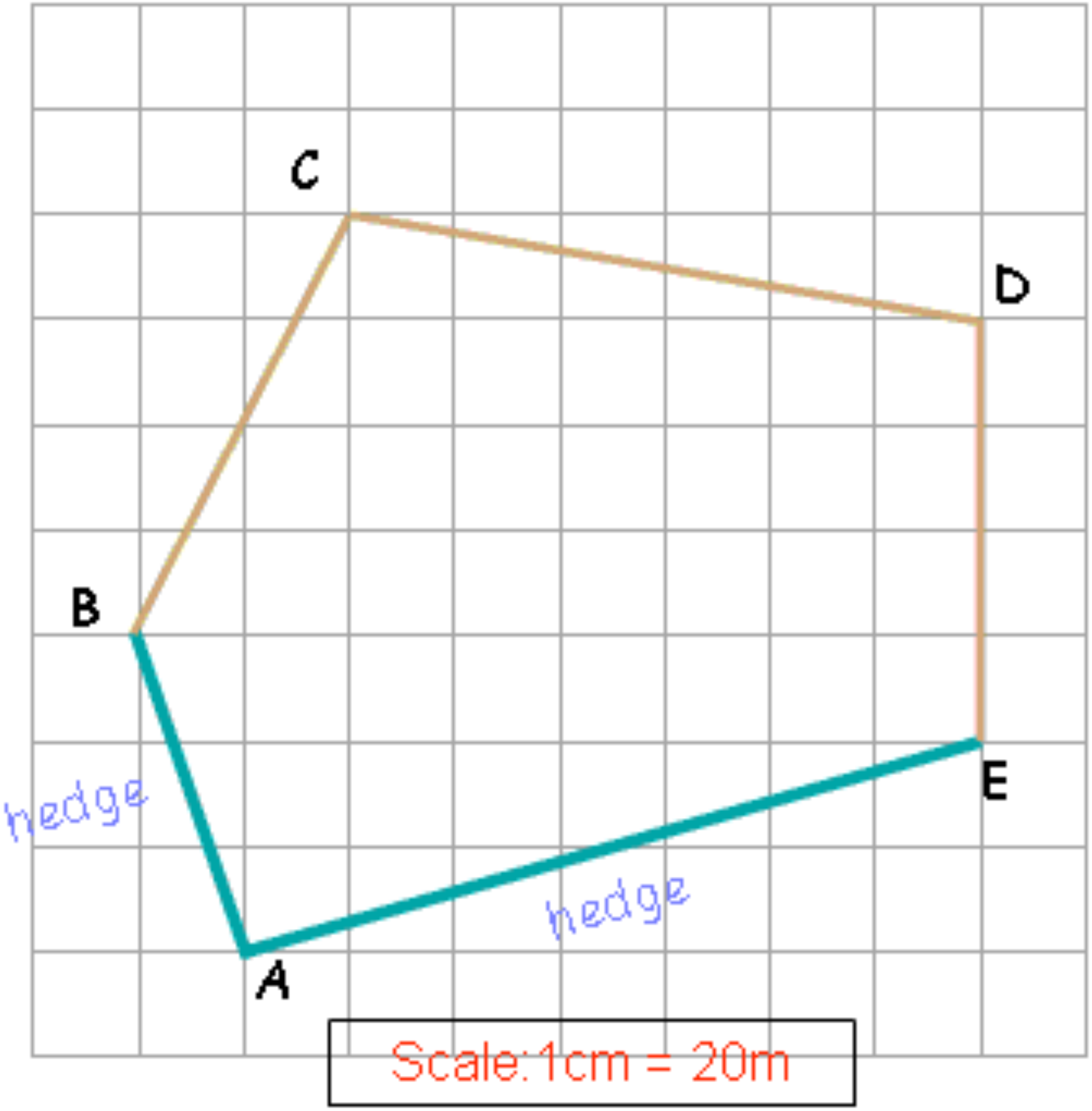
Example 2

A farmer wants to lay a water pipe across his field so that it is equidistant from two hedges. He also wants to connect a sprinkler in the exact centre of the pipe, that waters the field for 40 metres in all directions.

Skills needed: bisecting angles and bisecting lines



(a) Show the position of the pipe inside the field.

(b) Mark the point of connection for the sprinkler.

(c) Show the area of the field that is watered by the sprinkler.

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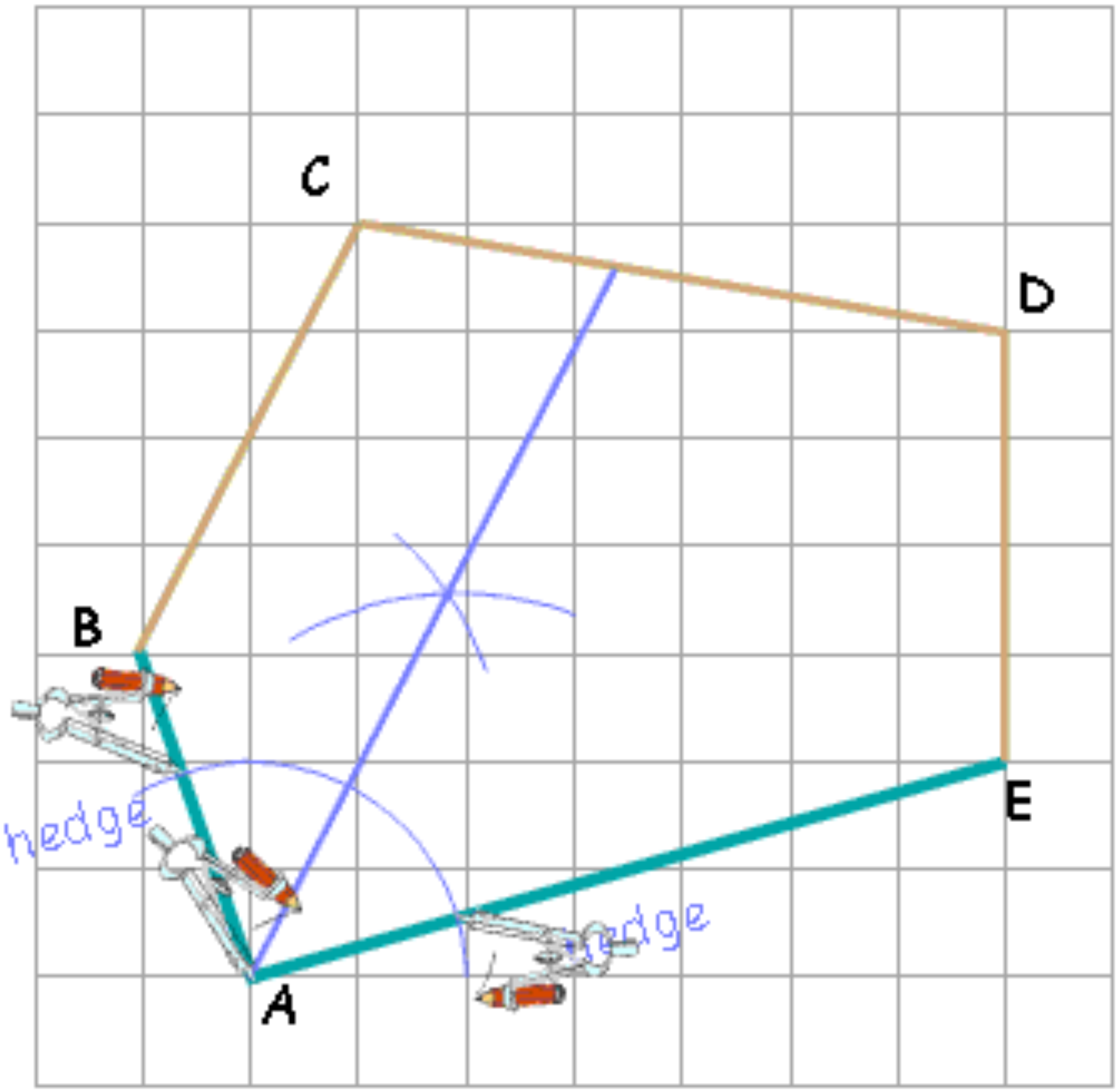

(a) Show the position of the pipe inside the field.

Steps:

1. Firstly, we need to realise what the question is asking... the pipe must always be the same distance from line AB as line AE... well, the only way to do that is to bisect the angle at A!

2. Place the pointy bit of your compass at A and mark a point on AE and AB

3. Now place your pointy bit on each of these new points and draw two arcs in the centre of the shape

4. Mark a new point where these two arcs cross

5. Draw a line that starts at A and goes through this crossing point and voila!... There is your pipe!

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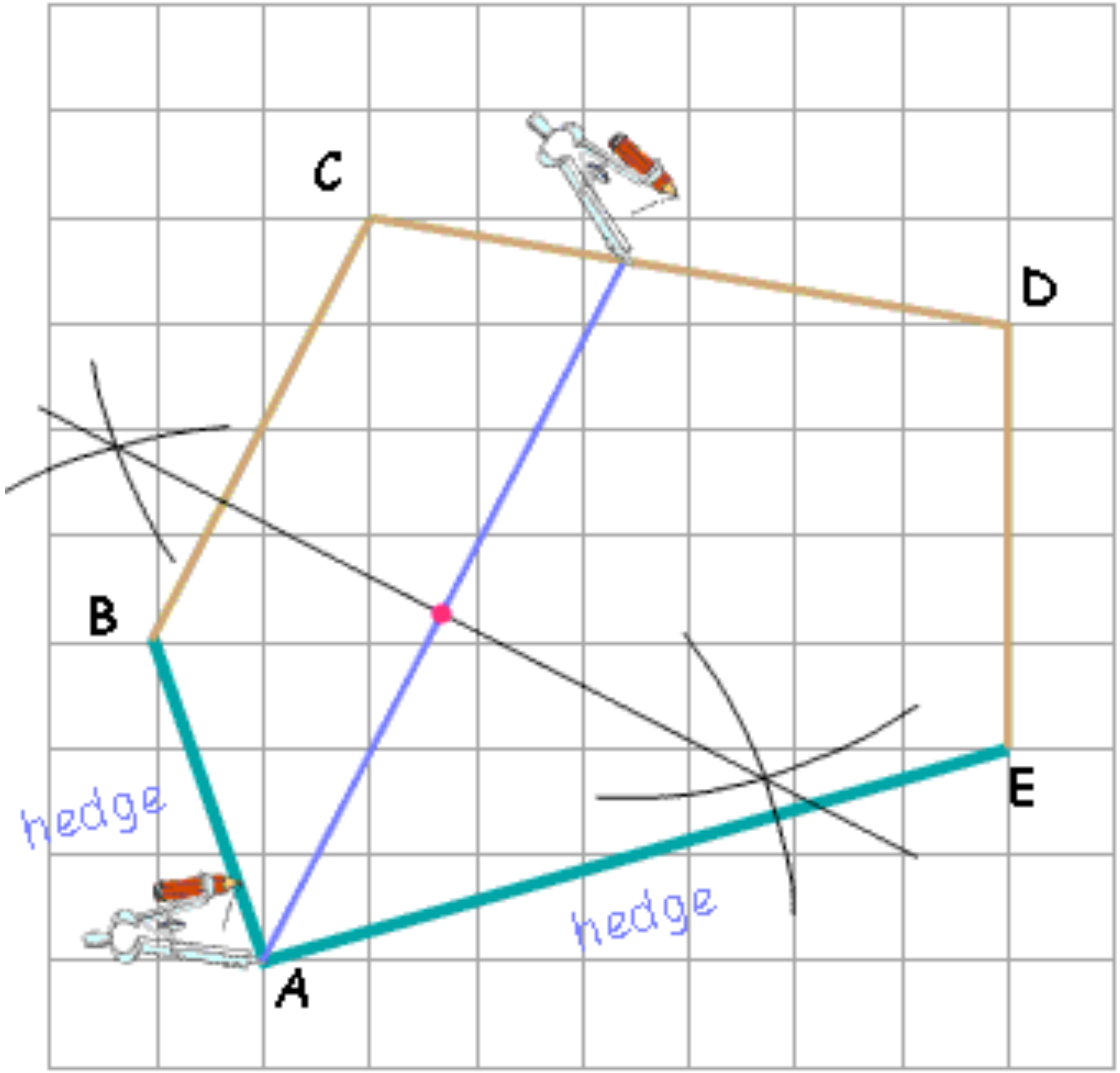

(b) Mark the point of connection for the sprinkler.

Steps:

1. Okay, so we have to find the exact centre of the pipe. Now it might be tempting to try to do it with your ruler… but that's no fun, and more importantly, it's not accurate! Instead, we must bisect the line
2. Place the pointy bit of your compass at A draw an arc on the right and an arc on the left
3. Place the pointy bit of the compass at the other end of the pipe and do the same.
4. Mark two points where these arcs cross
5. Draw a line through the two crossing points and where it hits the pipe is the exact centre!

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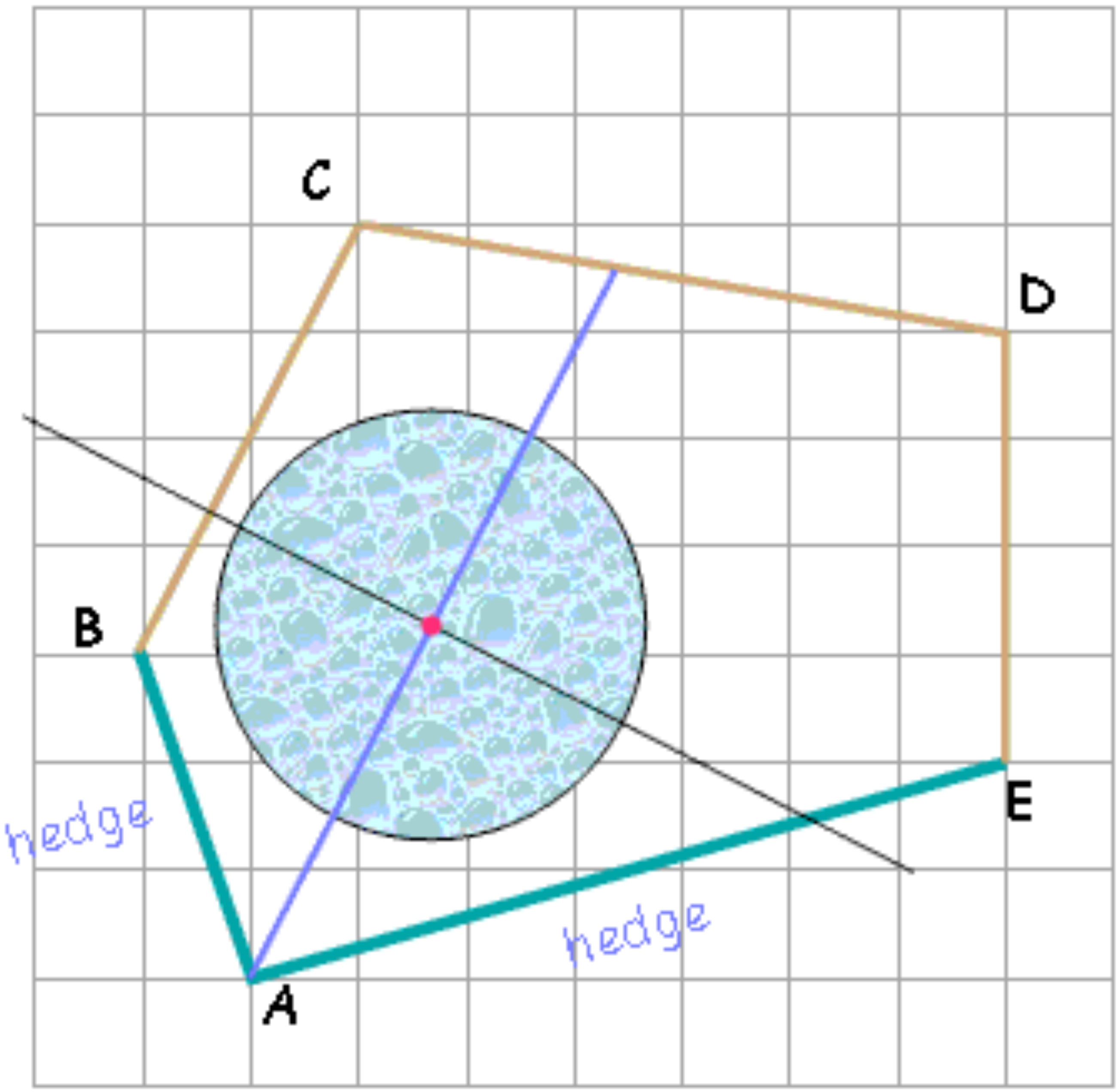


Scale: 1cm = 20m

(c) Show the area of the field that is watered by the sprinkler.

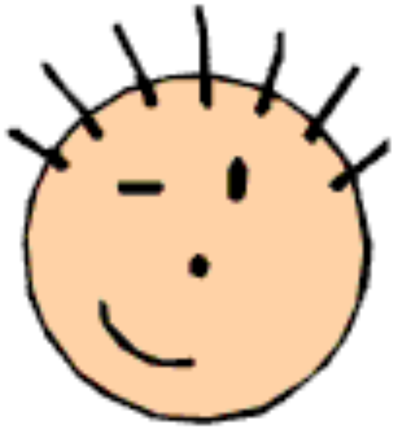
Steps:

1. First we must check our scale... 1cm = 20m, and we want to water 40m... so that is 2cm on our drawing!

2. The water can travel 2cm in all directions, so we must draw a circle

3. Place the pointy bit of the compass at the centre of the pipe and draw a circle with radius 2cm

4. Shade in the circle and you are done!



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5. Area

A Quick Word about Area

- Working out the areas of shapes is easy… so long as you remember the formulas!
- Sometimes you will be given them in exams, but more often they need to be fixed in your head!

NEVER FORGET every time you work out an area, give your answer as SQUARED UNITS
e.g. m^2, cm^2, km^2, mm^2 etc

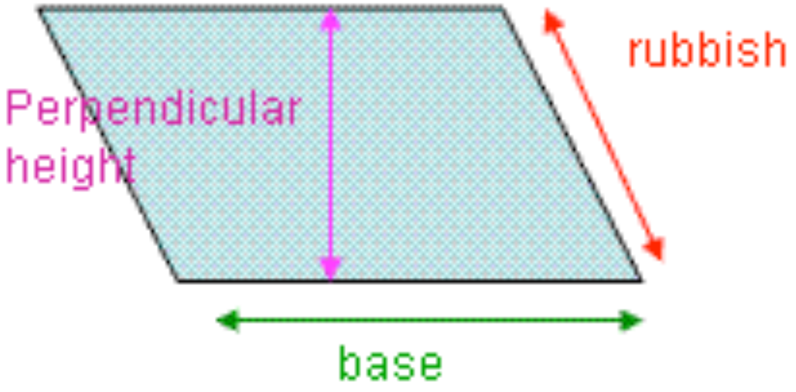
The Importance of Perpendicular Height

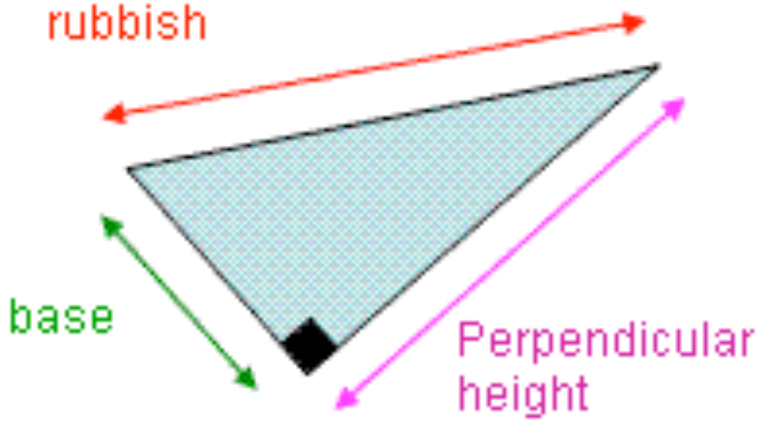
- As you will see, most of the formulas for area involve multiplying the base of the shape by it's height… but it's not just any old height!

- The height must be perpendicular to the base!

- What? All that means is that the height you measure must be at right angles (90^0) to the base

- So… if the base is horizontal (flat), then the height you want is vertical (straight up), not any slanted height that they may give you in the question to try and trip you up!





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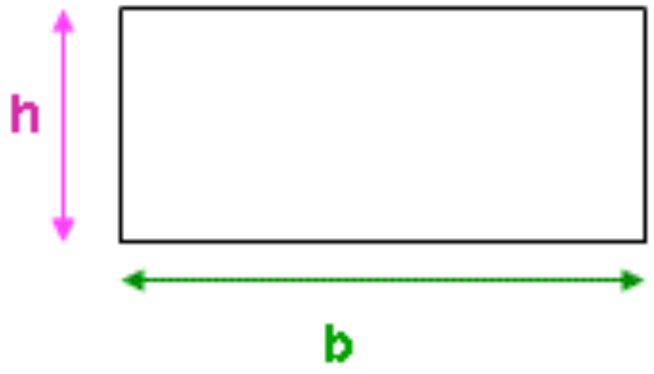
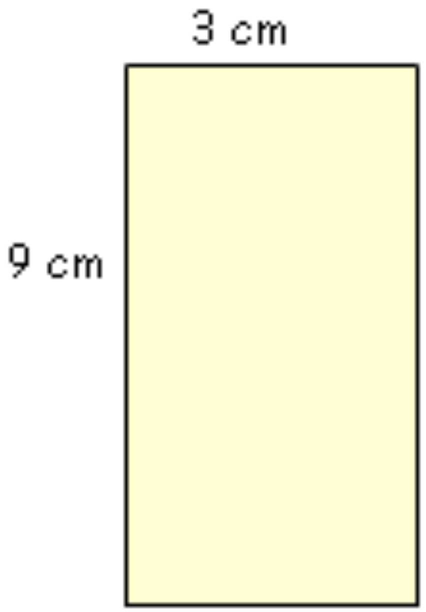
1. Rectangle



Area = $b \times h$

What to do: Multiply the base by the height!

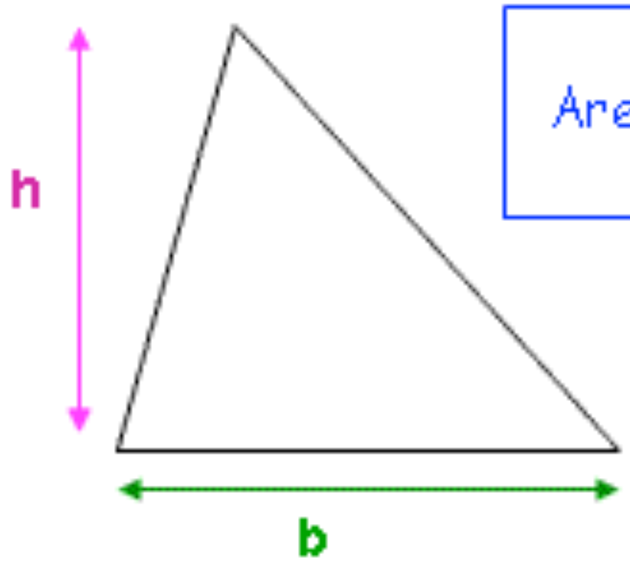
Example



Area = 9×3

= 27cm²

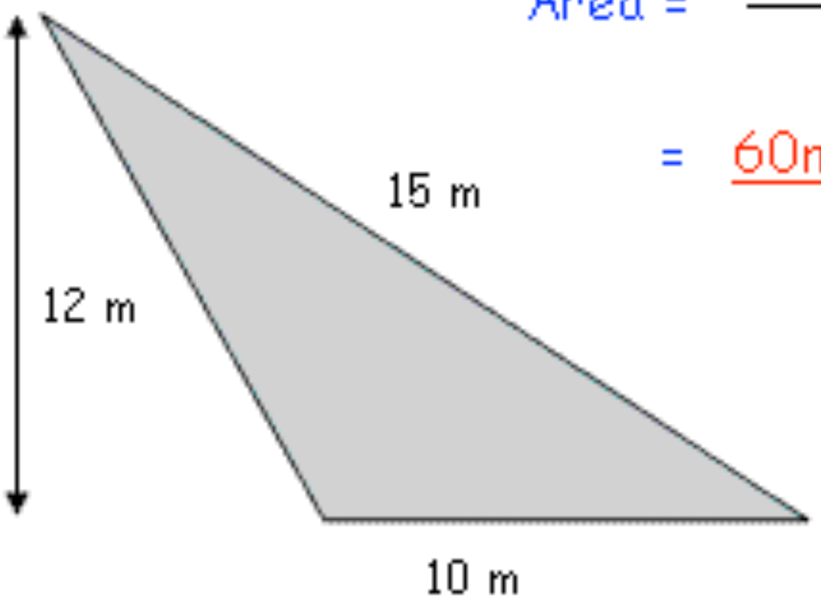
2. Triangle



$$\text{Area} = \frac{b \times h}{2}$$

What to do: Multiply the base by the (perpendicular) height and remember to divide your answer by 2!

Example



$$\text{Area} = \frac{10 \times 12}{2}$$

$= 60\text{m}^2$

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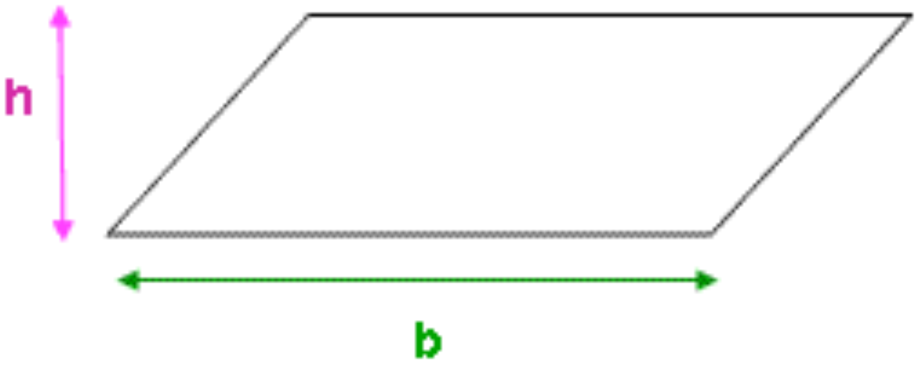
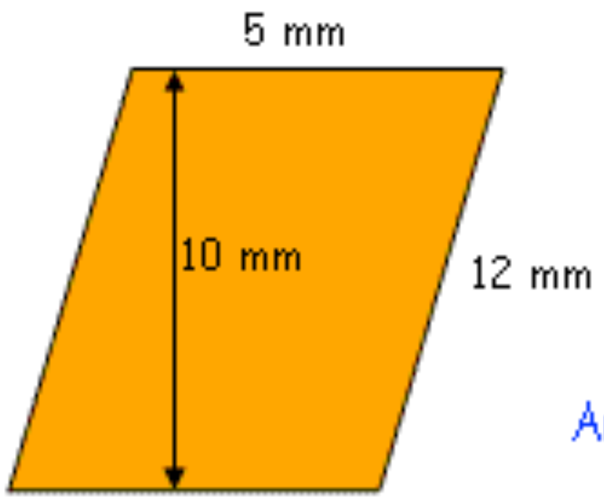
3. Parallelogram



Area = $b \times h$

What to do: Multiply the base by the perpendicular height… definitely not the slanted height!

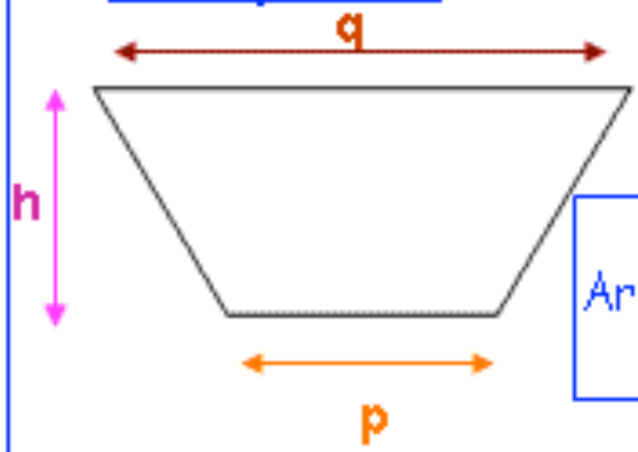
Example



Area = 5×10

= 50mm^2

4. Trapezium



Area = $\left(\frac{p + q}{2}\right) \times h$

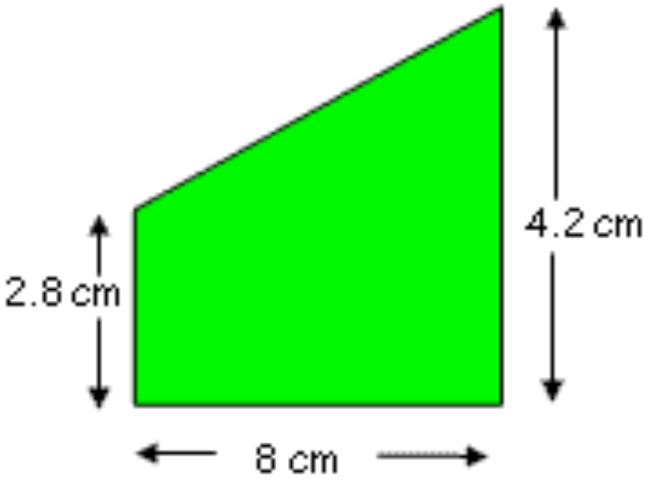
What to do: Add together the lengths of your two parallel sides and divide the answer by 2. This gives you the average length of your base. Then multiply this by the vertical height!

Example

Area = $\left(\frac{2.8 + 4.2}{2}\right) \times 8$

$= 3.5 \times 8$

= 28cm^2



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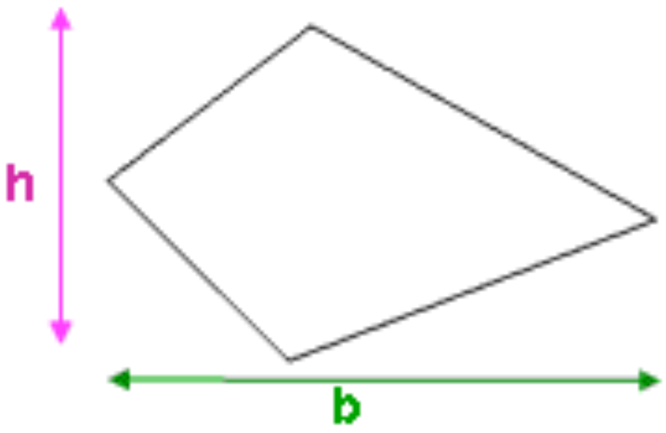
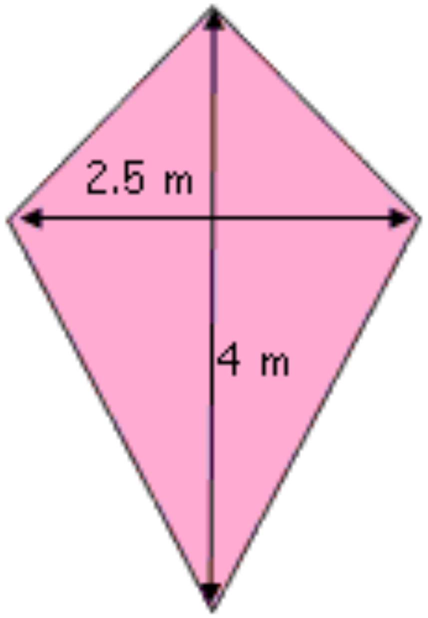
5. Kite



Area = $b \times h$

What to do: The base and height in a kite are just the two diagonals from point to point… so multiply them together!

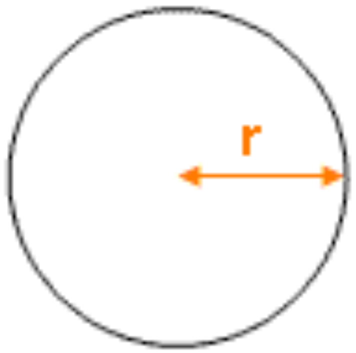
Example



Area = 2.5×4

= $10m^2$

6. Circle



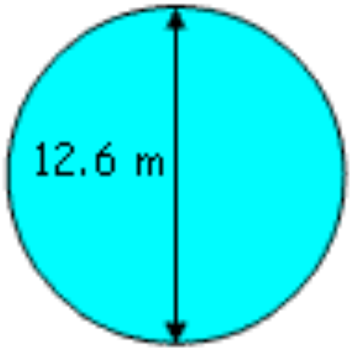
Area = $\pi \times r^2$

What to do: Find the radius of your circle (if you are given the diameter, just halve it!). Square the radius, and multiply your answer by pi!

Example

Diameter = 12.6 m

Radius = 6.3 m



Area = $\pi \times 6.3^2$

$= \pi \times 39.69$

= $124.7\ m^2$ (1dp)

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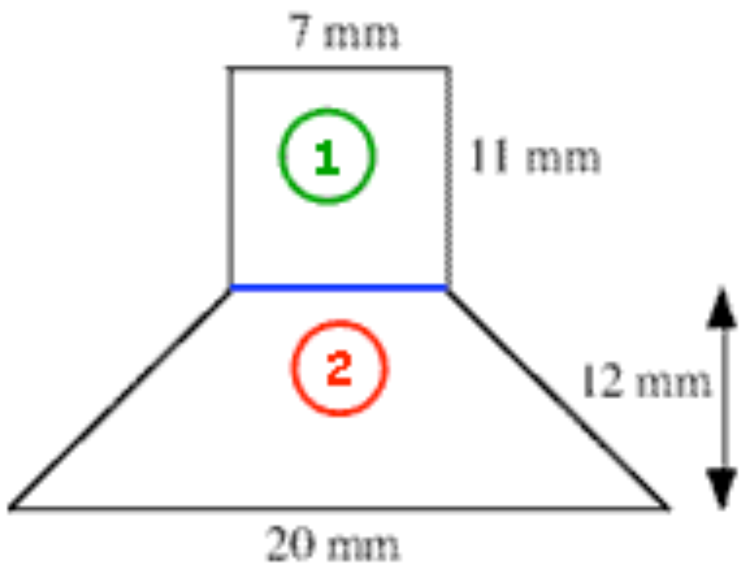
Compound Area

- Sometimes you are given quite complicated shapes and asked to work out the area.
- The technique here is to split them up into some of the 6 shapes you know how to work out the area of and just add together your answers!
- Try to be as clear as you can in your working to keep Mr Examiner happy!



I have chosen to split this shape up into a rectangle and a trapezium. It is also possible to split it up into rectangles and triangles. It is completely up to you!

① Rectangle

$$\boxed{\text{Area} = b \times h}$$

$$\text{Area} = 7 \times 11 \quad = \underline{77\text{mm}^2}$$

② Trapezium

$$\boxed{\text{Area} = \left(\frac{p + q}{2}\right) \times h}$$

$$\text{Area} = \left(\frac{20 + 7}{2}\right) \times 12 \quad = \underline{162\text{mm}^2}$$

Total Area

77 + 162

= 239 mm²

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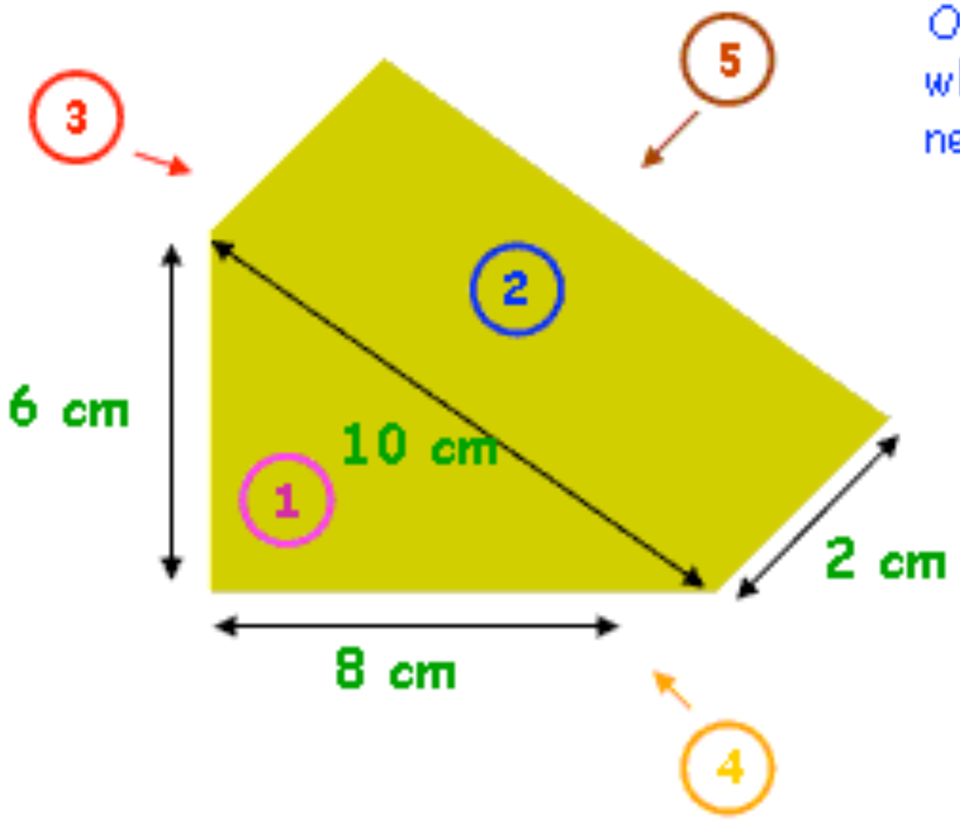
Surface Area

- I think of surface area as the exact amount of wrapping paper you would need to wrap up a 3D shape.
- People get themselves into a right muddle with surface area questions, mostly because they do not set them out properly and they end up forgetting sides or counting some twice!
- All you need to do is think about what flat 2D shape is on each side of your 3D object, work out its area, and tick off that side!
- It's just like compound area, only it gets you loads more marks!



Okay, so once again I am going to number each side, decide what shape it is, work out it's area, and then move onto the next!

① Triangle

$$\boxed{\text{Area} = \frac{b \times h}{2}}$$

$$\text{Area} = \frac{6 \times 8}{2}$$

$$= \underline{24\text{cm}^2}$$

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② Rectangle

$\boxed{\text{Area} = b \times h}$

Area = 2×10 = 20cm²

③ Rectangle

$\boxed{\text{Area} = b \times h}$

Area = 2×6 = 12cm²

④ Rectangle

$\boxed{\text{Area} = b \times h}$

Area = 8×2 = 16cm²

⑤ Triangle

Exact same shape as ①

Area = 24cm²

Total Area

24 + 20 + 12 + 16 + 24

= 96 cm²

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6. Volume

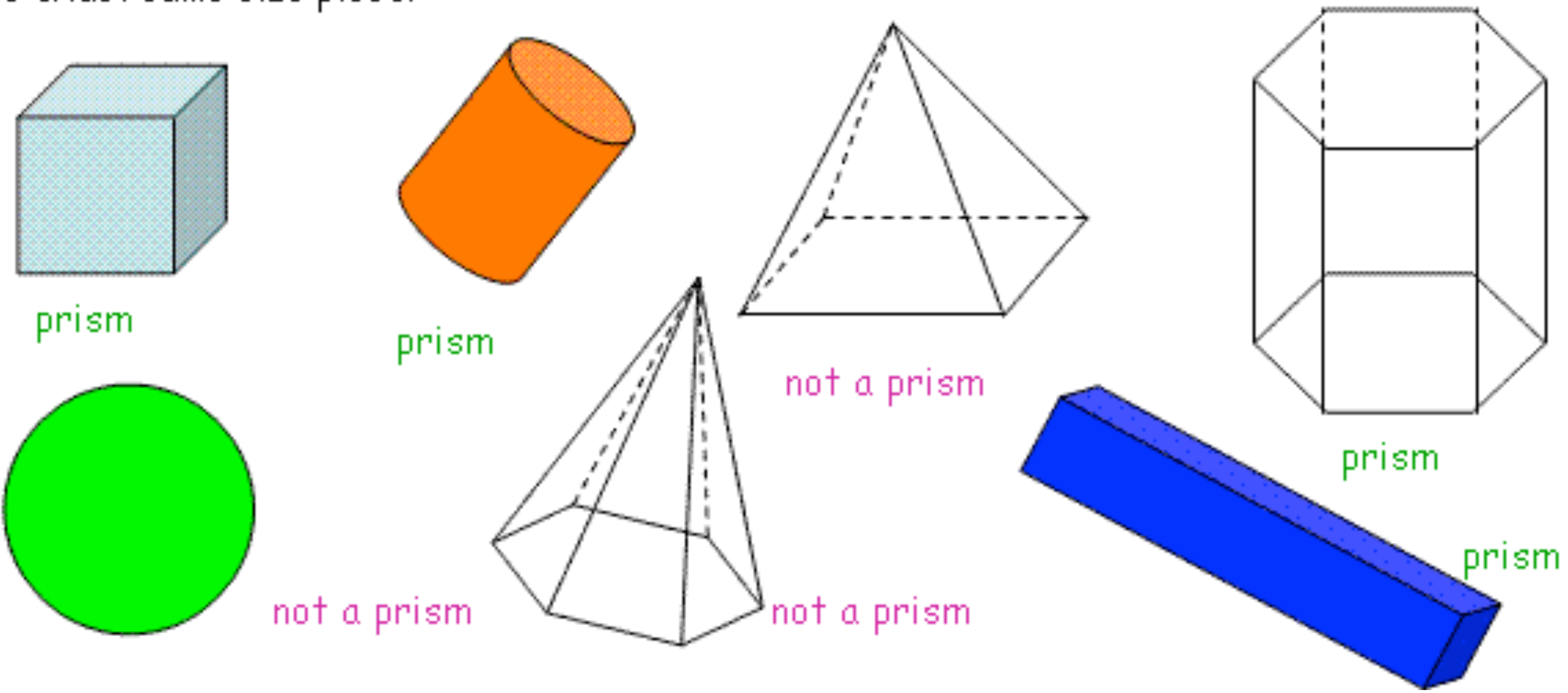
The Beauty of the Prism

Good News: So long as you know what a prism is, and you remember how to work out the areas of those 6 shapes we talked about in the last section (5. Area), you can do pretty much any volume question without needing any more formulas!... But remember your answers are UNITS CUBED!

What is a Prism?

A Prism is a 3D object whose face is the exact same shape throughout the object.

A birthday cake is the shape of a prism if it is possible to cut it in such a way to give everyone the exact same size piece!



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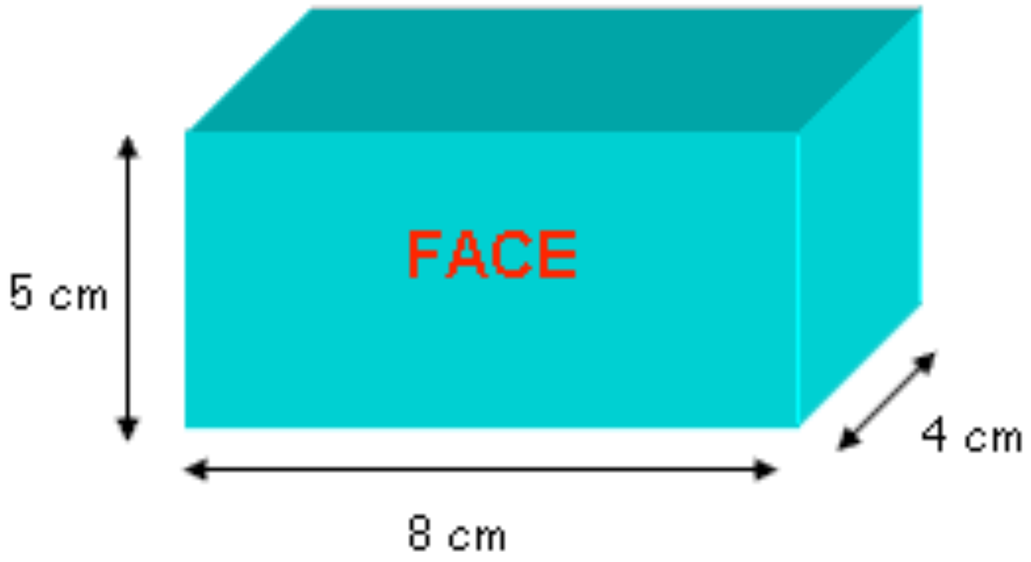
Working out the Volume of a Prism

So long as you can work out the area of the repeating face of the prism, the formula for the volume is the same for every single one:

Volume of a Prism = Area of Repeating Face x Length

Example 1 – Cuboid



Area of Repeating Face

Rectangle

$$\boxed{\text{Area} = b \times h}$$

$$\text{Area} = 8 \times 5 = \underline{40\text{cm}^2}$$

Volume of Prism

$$40 \times 4$$

$$= \underline{160\text{cm}^3}$$

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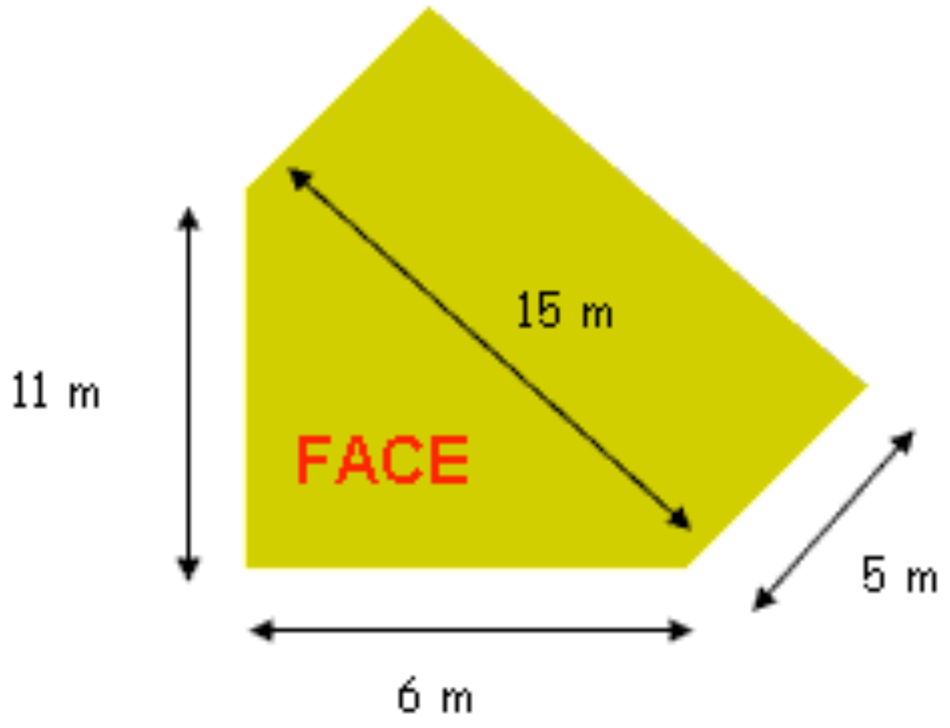
Example 2 – Triangular Based Prism



Area of Repeating Face

Triangle

$$\boxed{\text{Area} = \frac{b \times h}{2}}$$

$$\text{Area} = \frac{6 \times 11}{2}$$

$$= 33\text{m}^2$$

Volume of Prism

$$33 \times 5$$

$$= 165\text{m}^3$$

Note: Don't think you must use every measurement they give you. The 15m turned out to be pretty useless to us!

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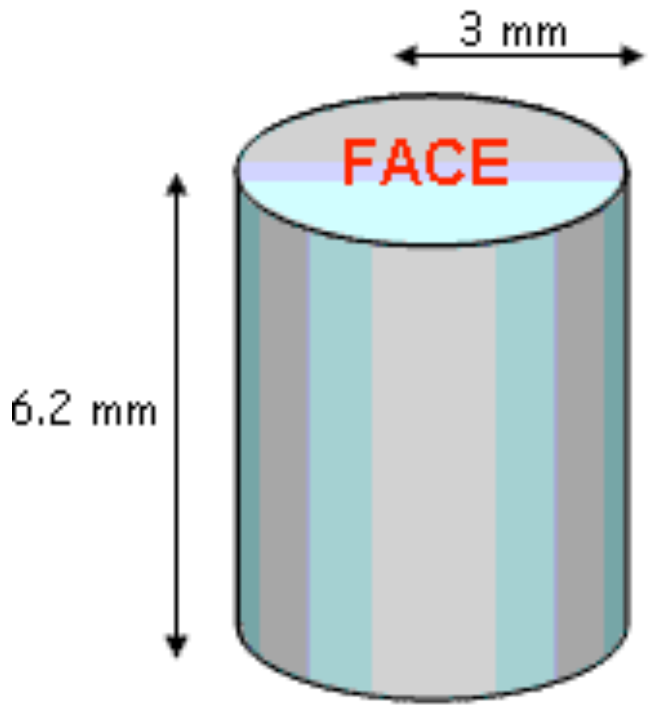
Example 3 – Cylinder



Area of Repeating Face

Circle

$$\boxed{\text{Area} = \pi \times r^2}$$

$$\text{Area} = \pi \times 3^2$$

$$= \pi \times 9$$

$$= \underline{28.274\ldots \text{ mm}^2}$$

Note: Keep this value in your calculator and use it for the next sum. It keeps your answer nice and accurate!

Volume of Prism

$$28.274\ldots \times 6.2$$

$$= \underline{175.3\text{mm}^3 \text{ (1dp)}}$$

Note: Sometimes "length" can mean "height" when you are working out the volume of the prism. It just depends which way the repeating face is facing!

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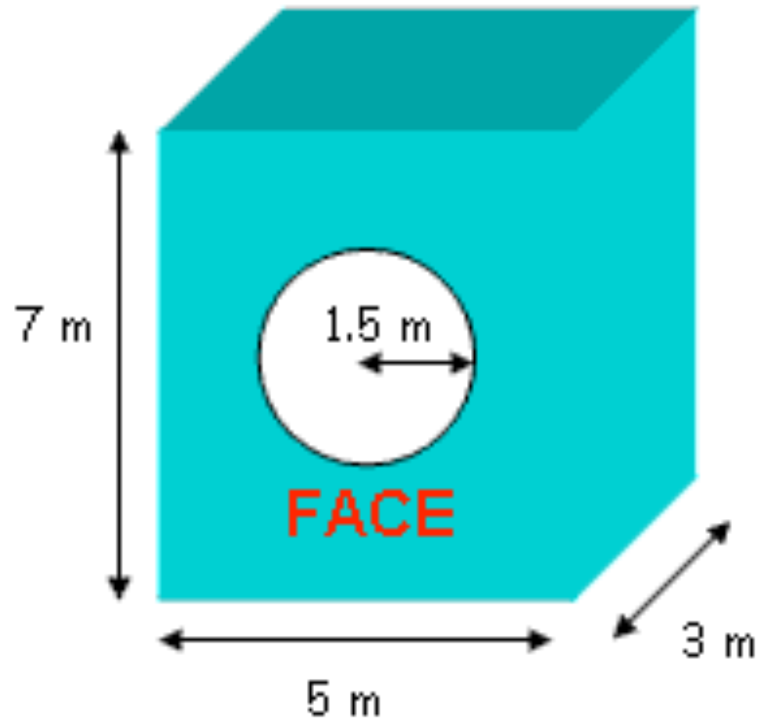
Example 4 – Complicated Prism

Note: This is still a prism as the front face repeats throughout the object!



Area of Repeating Face

This time it's a bit more complicated as we cannot work out the area of the face in one go. We must first work out the area of the complete rectangle, and then SUBTRACT the area of the missing circle to get our answer!

Rectangle

$$\boxed{\text{Area} = b \times h}$$

Area = 7×5

= 35m^2

Circle

$$\boxed{\text{Area} = \pi \times r^2}$$

Area = $\pi \times 1.5^2$

$= \pi \times 2.25$

= $7.068\ldots\ \text{m}^2$

Area of Repeating Face = $35 - 7.068\ldots$

= $27.931\ldots$

Volume of Prism

27.931×3

= 83.8m^3 (1dp)

Note: Try to avoid rounding in your working out by keeping the big numbers in the calculator, and then only round at the end!

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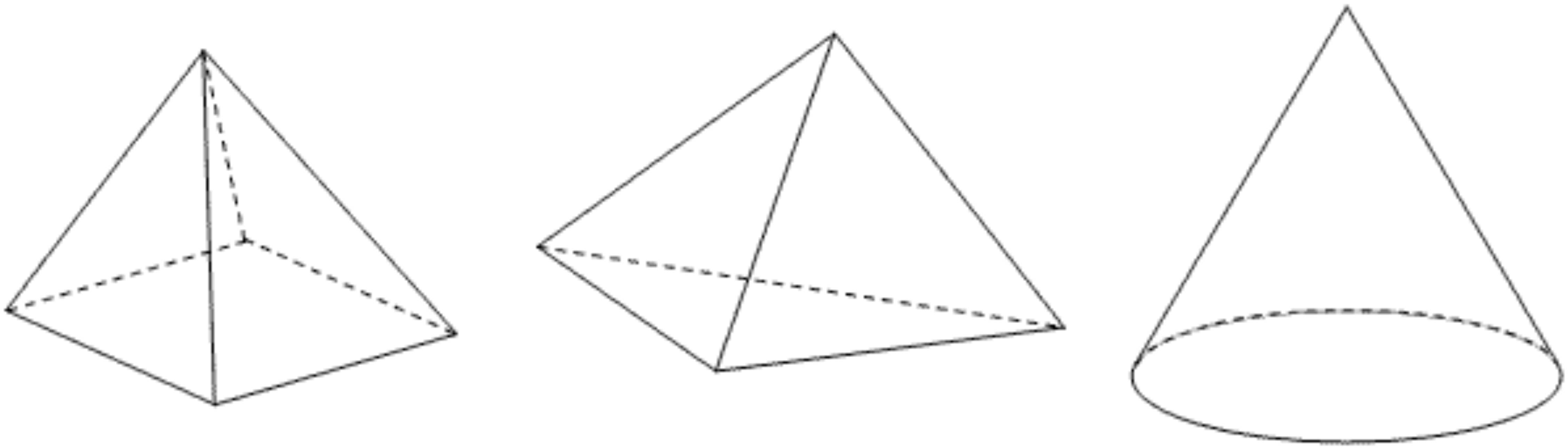
Working out the Volume of Pointy Shapes

Obviously, not all 3D shapes have a repeating face. Some shapes start off with a flat face and end up at a point. The technical name I have given to these shapes is... Pointy Shapes!



More Good News: Just like prisms, there is a general rule for working out the volume of all shapes like these:

$$\text{Volume of a Pointy Shape} = \frac{\text{Area of Face} \times \text{Length}}{3}$$

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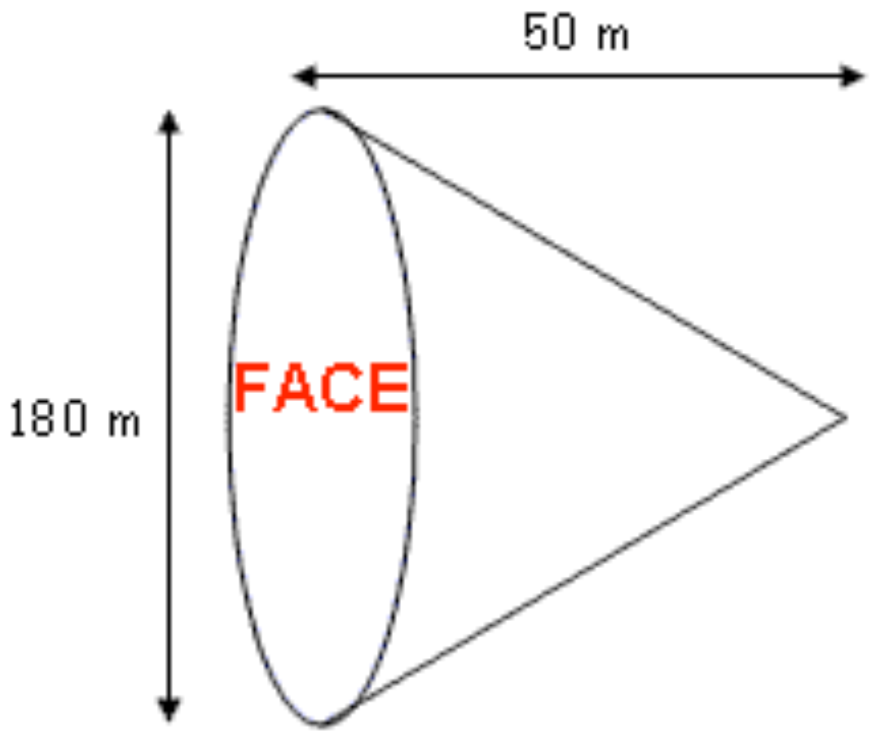
Example 4 – Cone



Area of Face

Circle

$$\boxed{\text{Area} = \pi \times r^2}$$

Diameter = 180m

Radius = 90 m

$$\text{Area} = \pi \times 90^2$$

$$= \pi \times 8100$$

= 25,446.9… m^2

Note: Keep this value in your calculator and use it for the next sum. It keeps your answer nice and accurate!

Volume of Pointy Shape

$$\frac{25,446.9\ldots \times 50}{3}$$

= 424,115 m^3 (nearest whole number)

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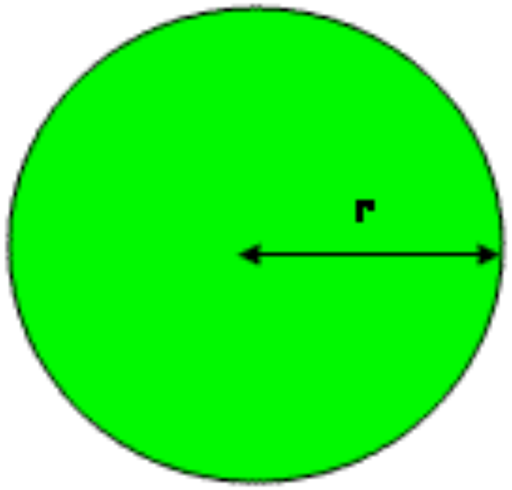
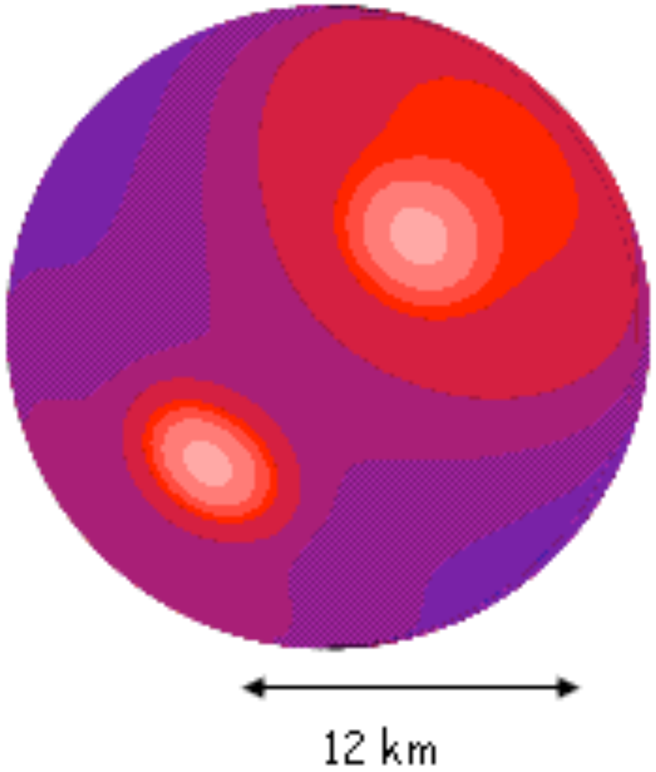
Example 5 – Sphere

Spheres do not have a repeating face, and they do not end in a pointy bit, so they have a rule all to themselves, and here it is...



Volume of a Sphere $= \frac{4}{3}\pi r^3$



Volume of Sphere

$$\frac{4}{3} \times \pi \times 120^3$$

$$= \frac{4}{3} \times \pi \times 1,728,000$$

$$= 7,238,229\ km^3$$

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7. Dimensions

What are Dimensions?

You may have heard people taking about dimensions in terms of objects:

One Dimension (1D)

Objects have just a LENGTH
Units of measurement include:
cm, mm, km, m, mile, etc

Two Dimensions (2D)

Objects have an AREA
Units of measurement include:
cm^2, mm^2, km^2, m^2, etc

Three Dimensions (3D)

Objects have a VOLUME
Units of measurement include:
cm^3, mm^3, km^3, m^3, etc

Four Dimensions (4D)

Objects exist in different times!
Fortunately we don't need to worry about this!

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The advantage of knowing this is that when we are given a formula, we can tell whether it is one for LENGTH, AREA, VOLUME, or just a load of rubbish!

Using Dimensions to Discover what Formulas are actually Working Out

Again, this is just my way of doing this, and feel free to bin it if you have a better one!

1. Change all the variables in the formula to the letter **D**

Note: Variables are just letters that represent lengths, widths and heights

2. Ignore all numbers (apart from powers!) and constants

Note: If a letter represents a constant instead of a variable, it will well you in the question

Remember: pi (π) is just a number!

3. You should now be left with an expression just containing D's, which you can use your algebra skills to simplify

Crucial: When you are simplifying, DO NOT cancel anything out! You'll see why in the examples!

4. Look at what you are left with. If the formula only contains...

D - this is a formula for length

D^2 - this is a formula for area

D^3 - this is a formula for volume

Any combination - this formula is rubbish!

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Examples

In all the following examples, l, w and h are variables representing lengths, and k is a constant

Determine whether these formulas calculate length, area, volume or nothing

1. $5wh$

1. Okay, so our variables are w and h, and they become **D**

$5DD$

2. Let's get rid of our number

DD

3. We only have D's left in our expression, so it's looking good! Now, let's use our algebra skills to simplify, remembering that in algebra the multiplication sign is disguised!

D^2

4. We are left with: D^2

Which means this is a formula for... AREA

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$\boxed{2.}$ $7h(l-w)+2w^2$

1. Okay, so our variables are w, l and h, and they become **D**

$$7D(D-D)+2D^2$$

2. Let's get rid of our numbers

$$D(D-D)+D^2$$

3. Now it's time to simplify... but be careful! It's fine to expand our brackets, but do not cancel anything out!

$$D^2-D^2+D^2$$

4. We are left with a formula that just contains: D^2

Which means this is a formula for... AREA

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3. $\frac{2}{3}h(lh+\pi w-h^2)$

1. Okay, so our variables are w, l and h, and they become **D**

2. Let's get rid of our numbers… remember, pi (π) is just a number, and so are fractions!

3. Now it's time to simplify… but be careful! It's fine to expand our brackets, but do not cancel anything out! I'm going to do this in two stages!

4. We are left with a formula that contains a mixture of:

D^2 and D^3

Which means this formula is a load of rubbish

$\frac{2}{3}D(DD+\pi D-D^2)$

$D(DD+D-D^2)$

$D(D^2+D-D^2)$

$\rightarrow D^3+D^2-D^3$

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4.

$$\frac{5h^3 + 2lw^2 - hlw}{6}$$

1. Okay, so our variables are w, l and h, and they become **D**

$$\frac{5D^3 + 2DD^2 - DDD}{6}$$

2. Let's get rid of our numbers...

$$D^3 + DD^2 - DDD$$

3. Now it's time to simplify... but be careful! We are definitely not going to cancel anything out!

$$D^3 + D^3 - D^3$$

4. We are left with a formula that only contains: D^3

Which means this formula is for volume

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5.

$$\frac{kl^3 + \pi h w^2}{8hl}$$

1. Okay, so our variables are w, l and h, and they become **D**

$$\frac{kD^3 + \pi DD^2}{8DD}$$

2. Let's get rid of our numbers... and our constant K!

$$\frac{D^3 + DD^2}{DD}$$

3. Now it's time to simplify... I'm going to simplify the terms on the top and bottom first, and then divide the top by the bottom!

$$\frac{D^3 + D^3}{D^2}$$

$$\longrightarrow D + D$$

4. We are left with a formula that only contains: D

Which means this formula is for length

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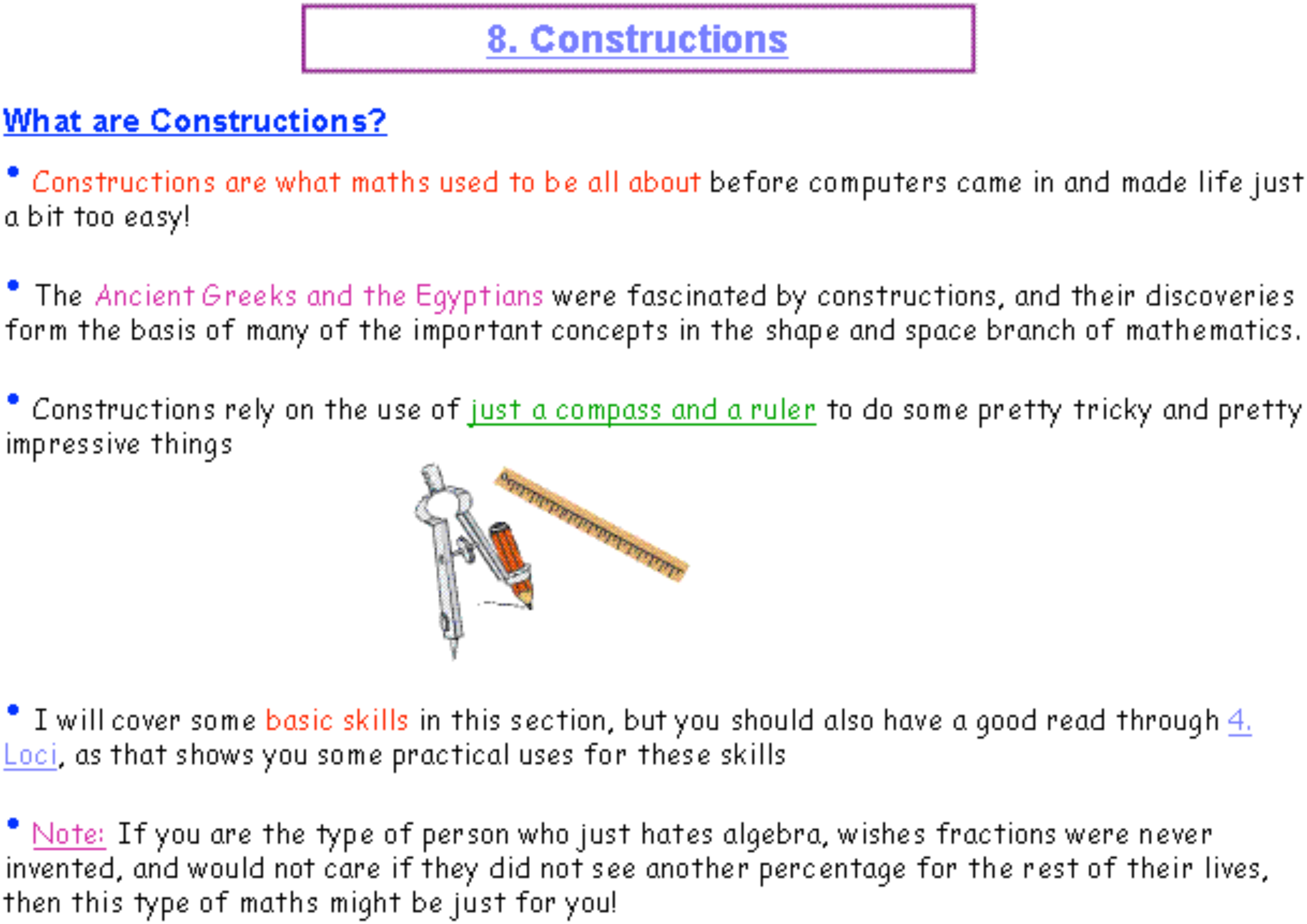

8. Constructions

What are Constructions?

- Constructions are what maths used to be all about before computers came in and made life just a bit too easy!

- The Ancient Greeks and the Egyptians were fascinated by constructions, and their discoveries form the basis of many of the important concepts in the shape and space branch of mathematics.

- Constructions rely on the use of just a compass and a ruler to do some pretty tricky and pretty impressive things

- I will cover some basic skills in this section, but you should also have a good read through 4. Loci, as that shows you some practical uses for these skills

- Note: If you are the type of person who just hates algebra, wishes fractions were never invented, and would not care if they did not see another percentage for the rest of their lives, then this type of maths might be just for you!

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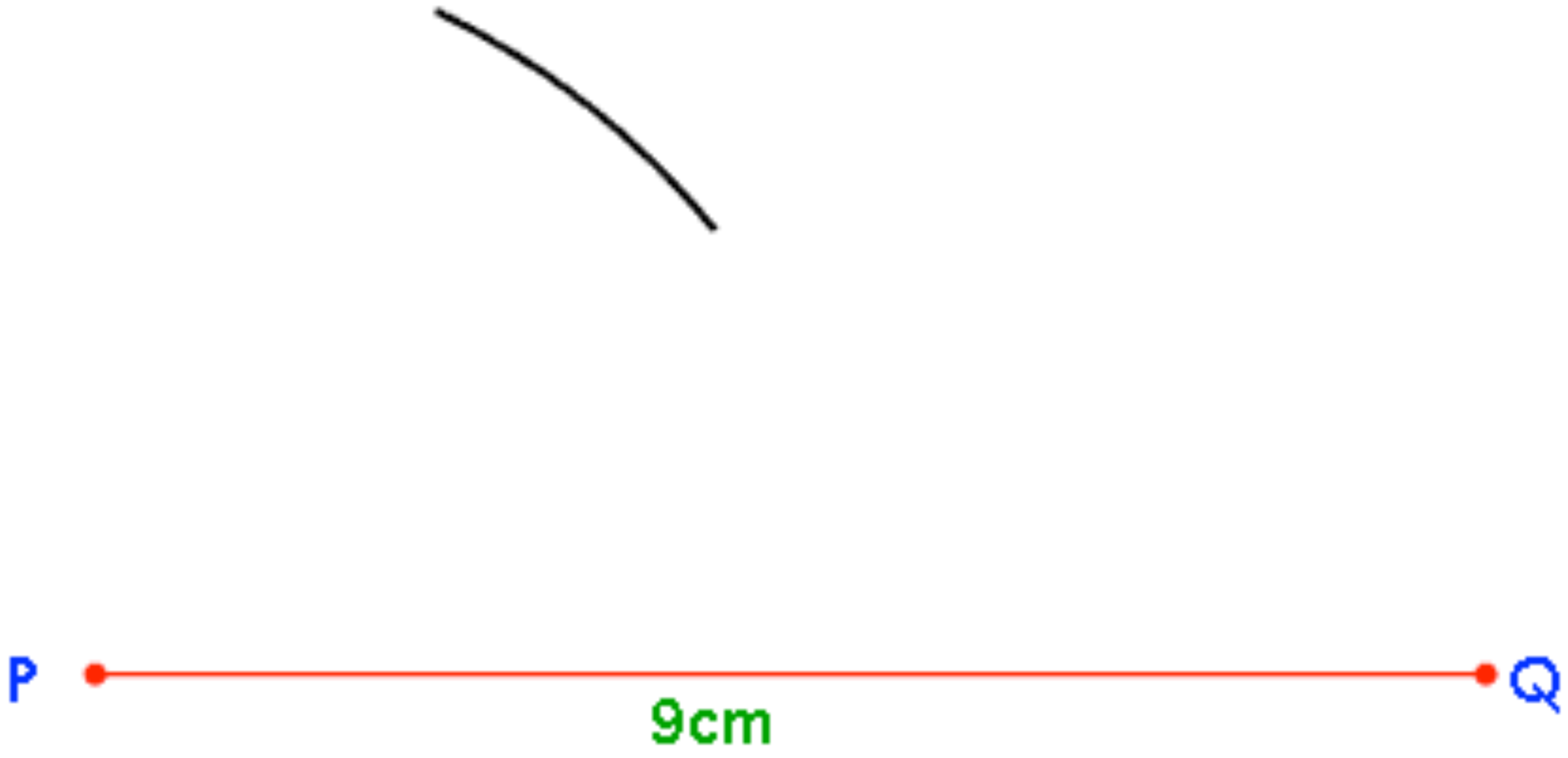
1. Drawing a Triangle Given 3 Sides

Construct the triangle PQR with sides: PQ = 18cm, PR = 10cm and QR = 14cm

1. Select the longest side as your base, and carefully draw a horizontal line 18cm long, labelling the ends **P** and **Q**

2. Set your compass to 10cm, place the pointy bit at **P**, and draw an arc:



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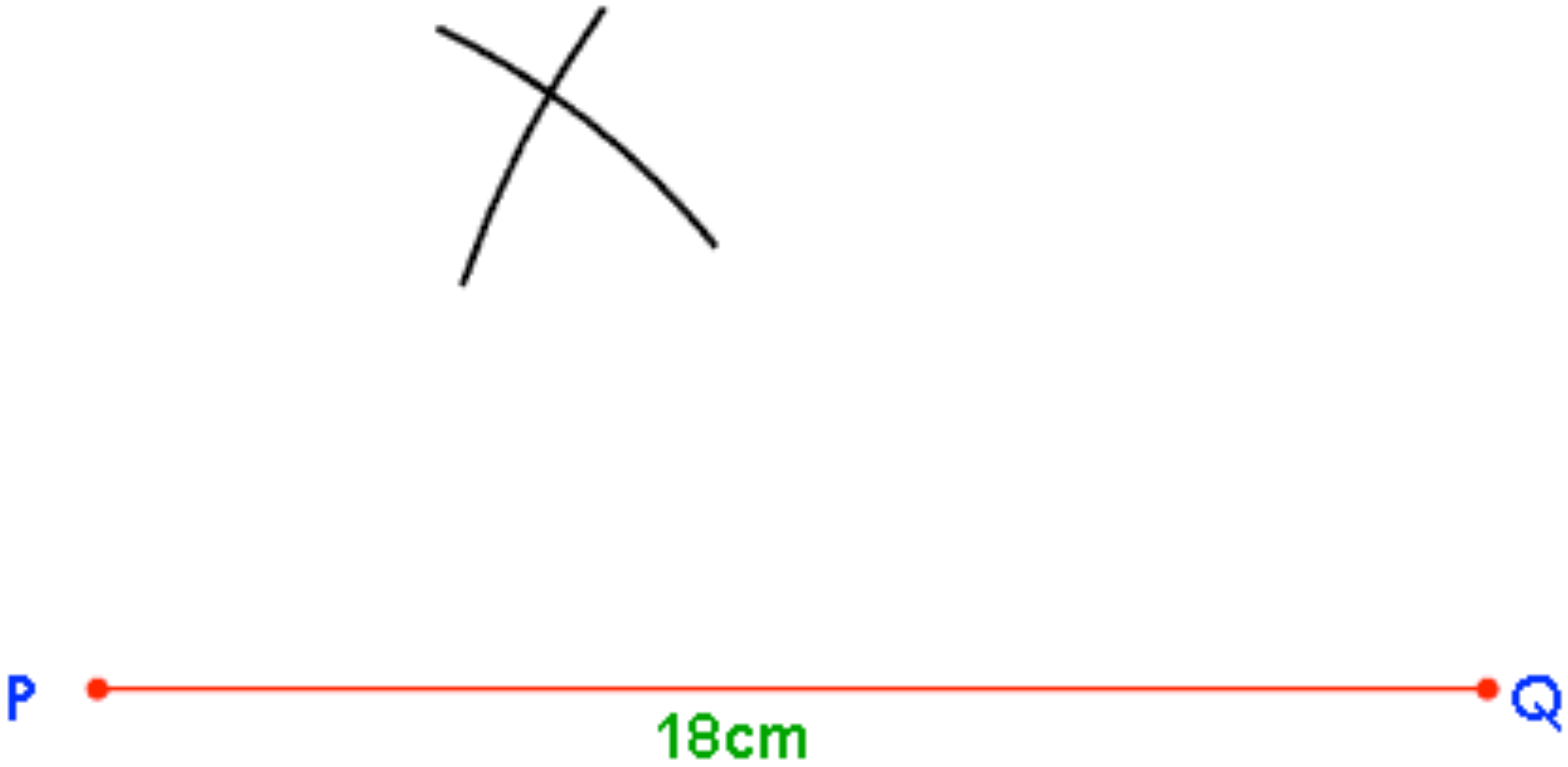
3. Now set your compass to 14cm, place the pointy bit at Q and draw and arc:



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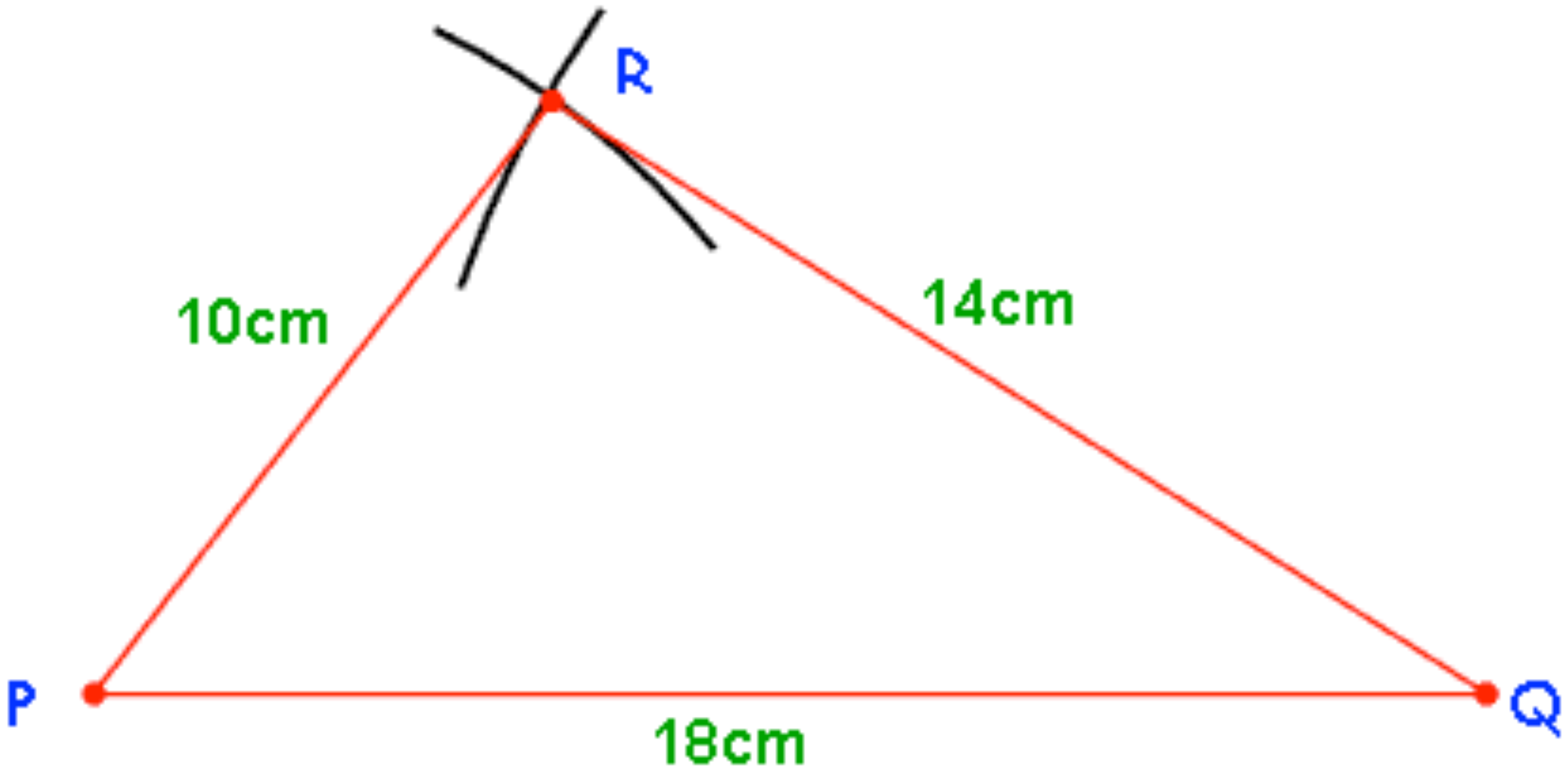
4. Label the point where these two arcs meet **R**, join up your lines, and check with a ruler that you have got your measurements correct!



NOTE: Never rub out your construction lines!

NOTE: If you wanted to construct an Equilateral Triangle, then whatever length you choose for your base, just make sure you set your compass to the exact same length for both of your arcs!

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2. Drawing a Perpendicular Bisector

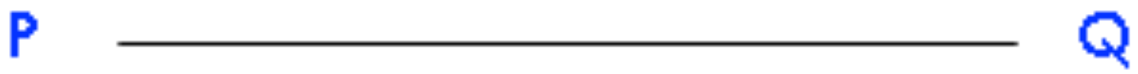
What does that mean?

Perpendicular: At right angles (90^0)

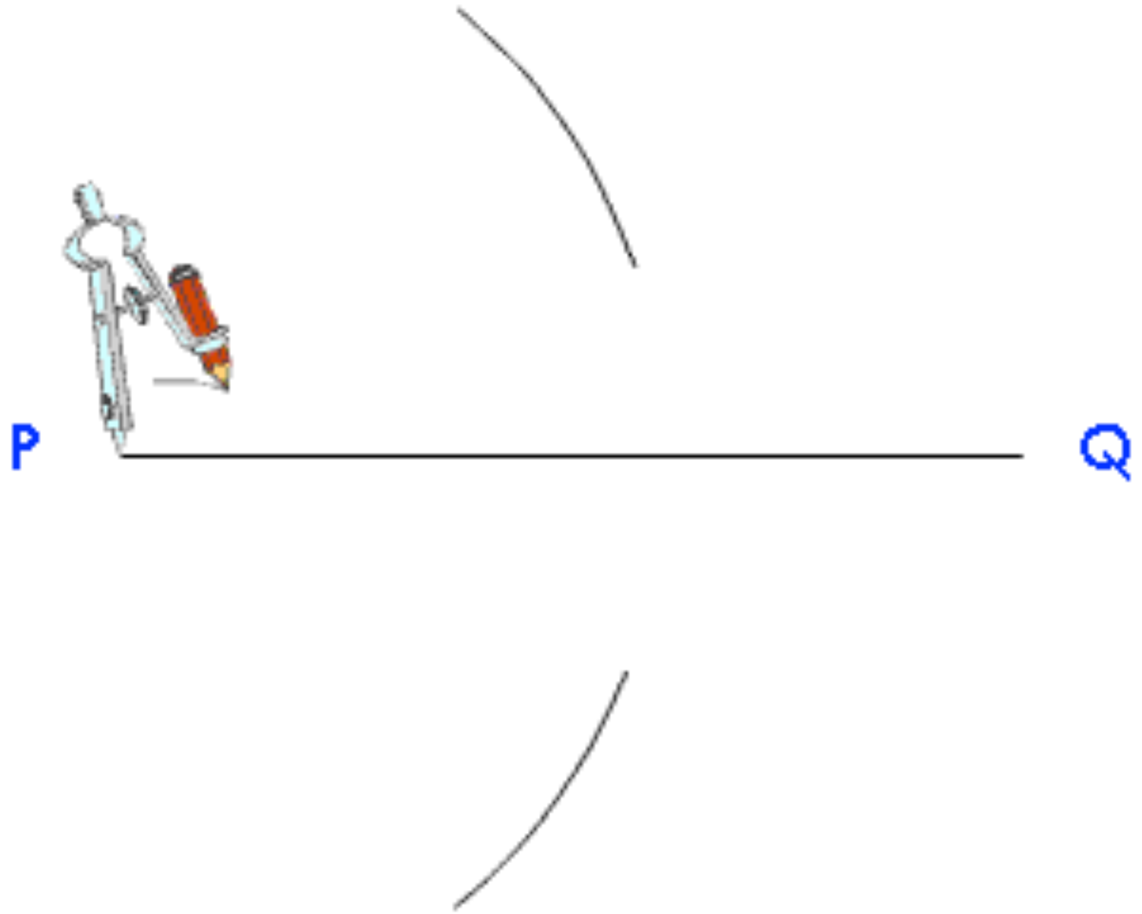
Bisector: Chop in half

So… if we are given a line to start with, we want a line that chops it in half at right

Construct a perpendicular bisector to the line PQ



1. Set your compass to over half the length of the line. Place the pointy bit of the compass at **P** and draw an arc above and below the line:



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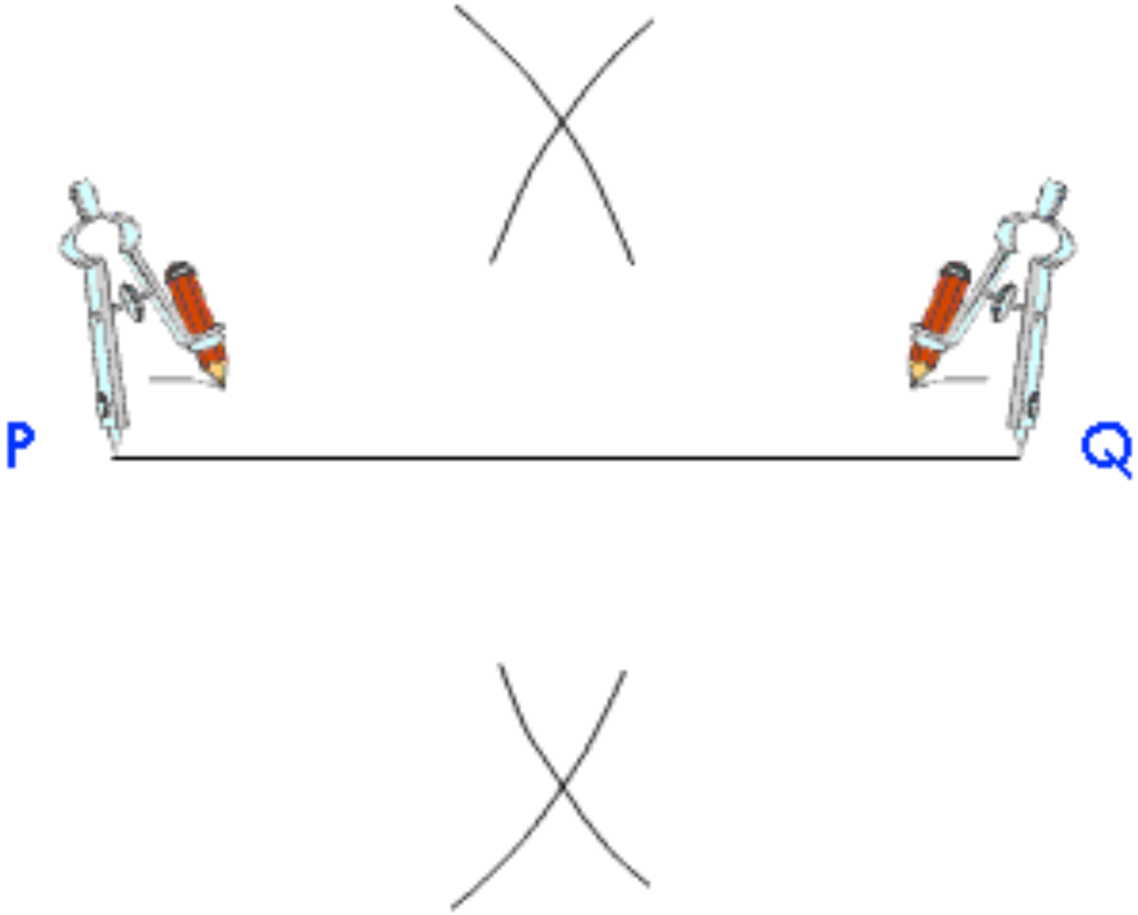
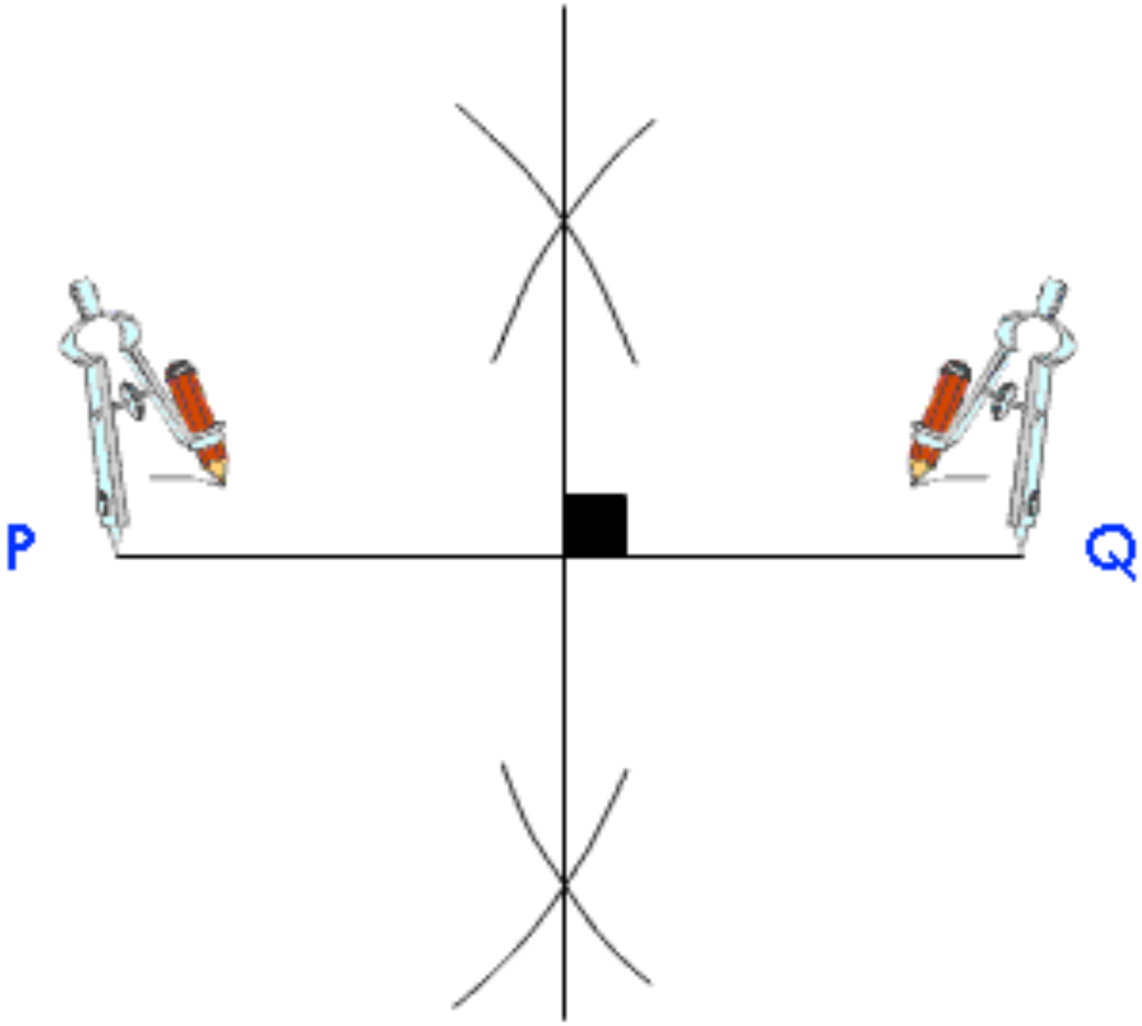


2. Making sure you keep your compass at the exact same setting, place the pointy bit at **Q** and draw two more arcs.

3. With your ruler, draw a straight line through the two points where the arcs cross, and that is your perpendicular bisector!



Note: Every point on this new line is the exact same distance from point P as it is from point Q!

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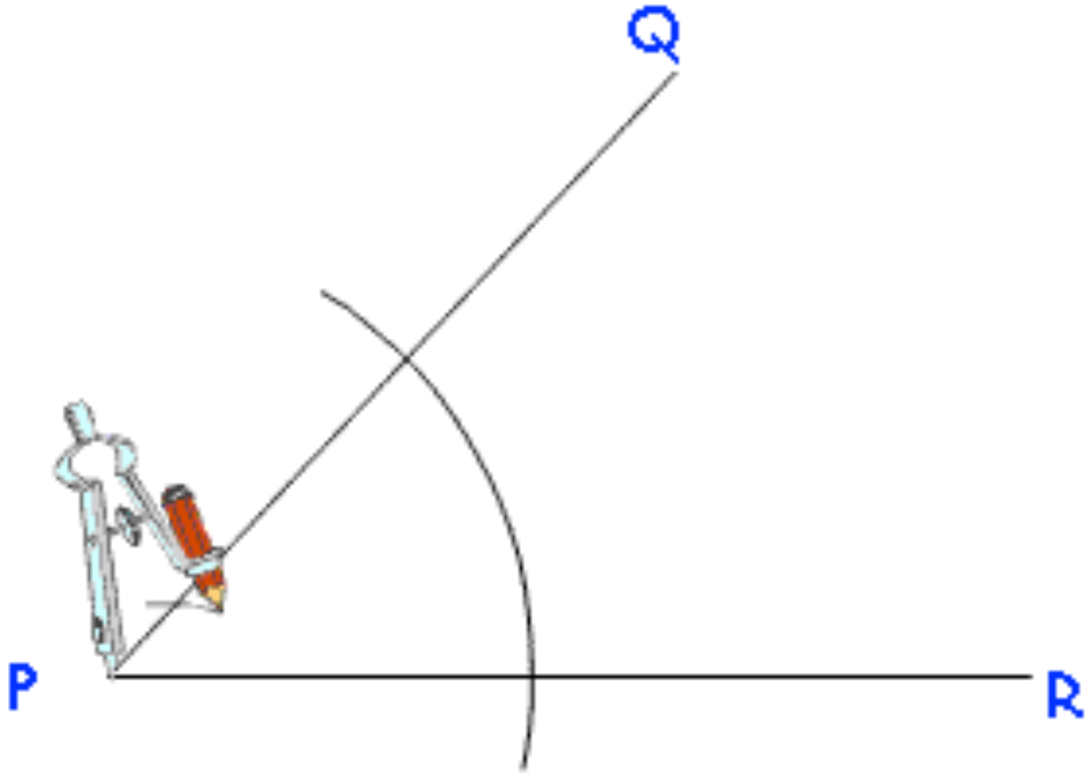
3. Drawing an Angle Bisector

What does that mean?

Bisector: Chop in half

So… if we are given an angle, we need to chop it in half… without using an angle measurer!

Construct an angle bisector for the angle made by lines PQ and PR



1. Place the pointy bit of your compass at **P** and draw an arc which crosses lines **PQ** and **PR**

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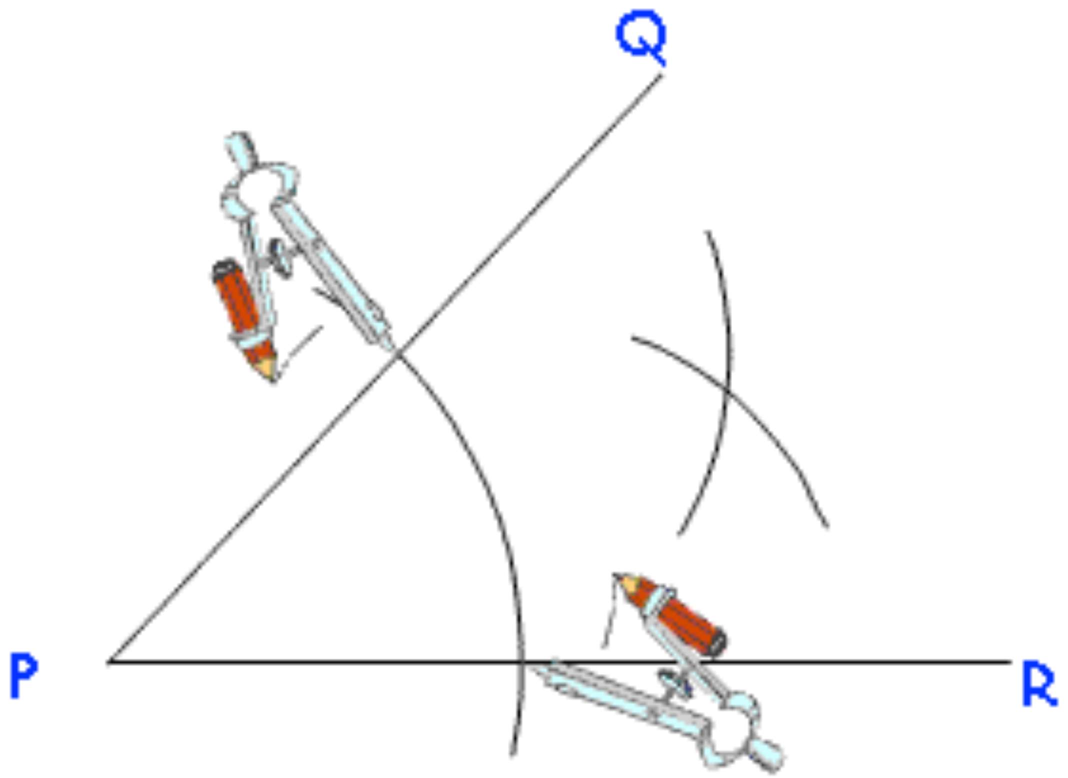
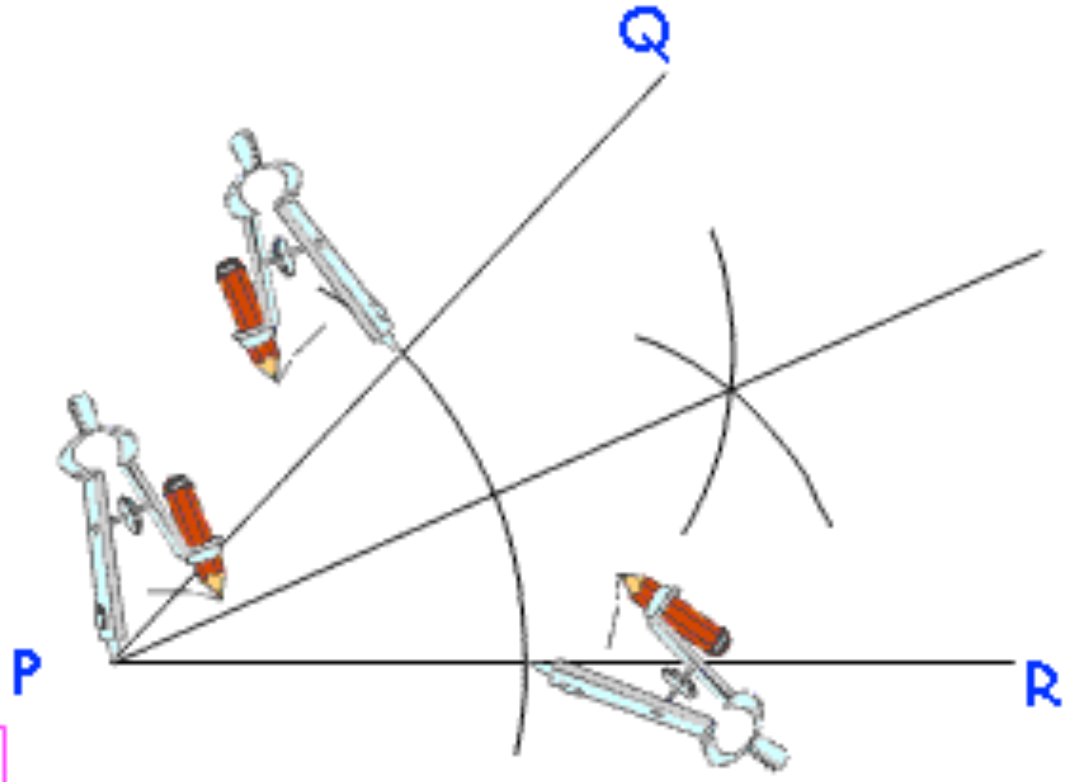


2. Place the pointy bit of the compass at both of the places where the arc hits the lines and draw two arcs

Crucial: You must not change the setting of the compass at this stage!

3. With your ruler, draw a straight line from **P** through the intersection of the arcs.

This is your angle bisector!



Note: Every point on this new line is the exact same distance from line PQ as it is from line PR!

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9. Vectors

1. What are Vectors?

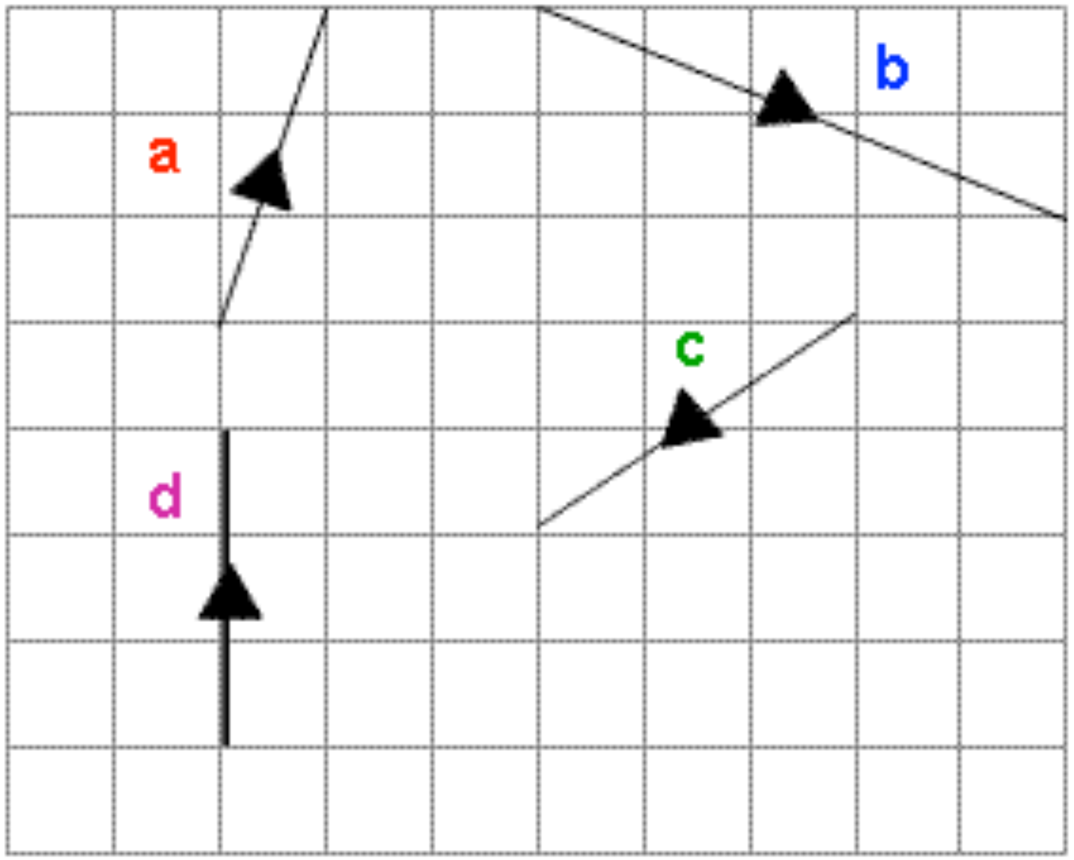
- Vectors are just a posh (and quite convenient) way of describing how to get from one point to another

- Starting from the tail of the vector, the number on the top tells you how far right/left to go, and the number on the bottom tells your how far up/down

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

If this number is positive, you move right, if it is negative, you move left

If this number is positive, you move up, if it is negative, you move down



a	$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	1 to the right, and 3 up
b	$\begin{pmatrix} 5 \\ -2 \end{pmatrix}$	5 to the right, and 2 down
c	$\begin{pmatrix} -3 \\ -2 \end{pmatrix}$	3 to the left, and 2 down
d	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	0 to the right, and 3 up

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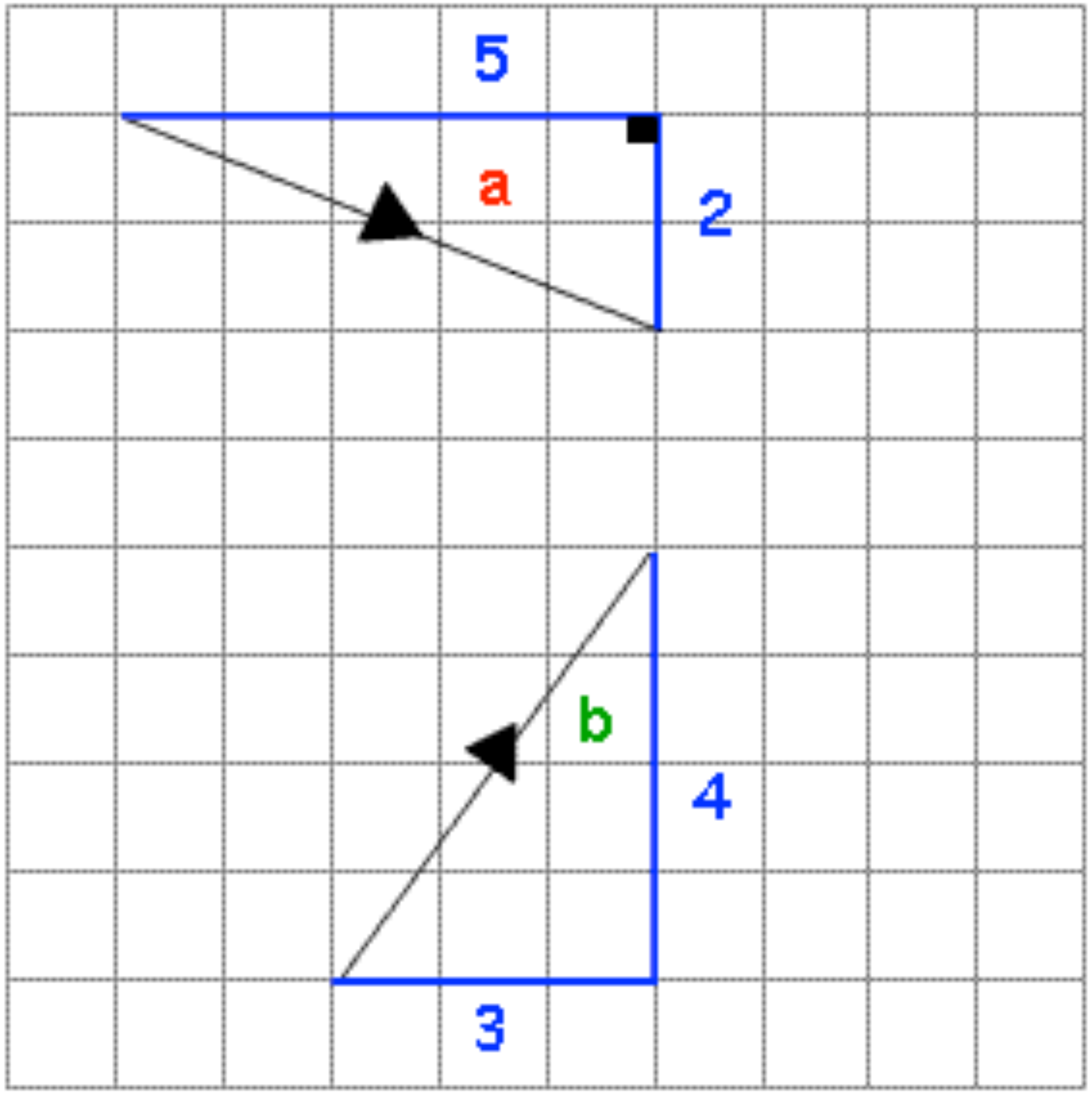
2. The Magnitude of Vectors

- By forming right-angled triangles and using Pythagoras' Theorem, it is possible to work out the magnitude (size) of any vector



a $\begin{pmatrix}5\\-2\end{pmatrix}$

$$a^2 = 5^2 + 2^2$$
$$a = \sqrt{5^2 + 2^2}$$
$$a = 5.4 \ (1 \text{ dp})$$

b $\begin{pmatrix}3\\4\end{pmatrix}$

$$b^2 = 3^2 + 4^2$$
$$b = \sqrt{3^2 + 4^2}$$
$$b = 5$$

Note: Because you are squaring the numbers, you do not need to worrying about negatives!

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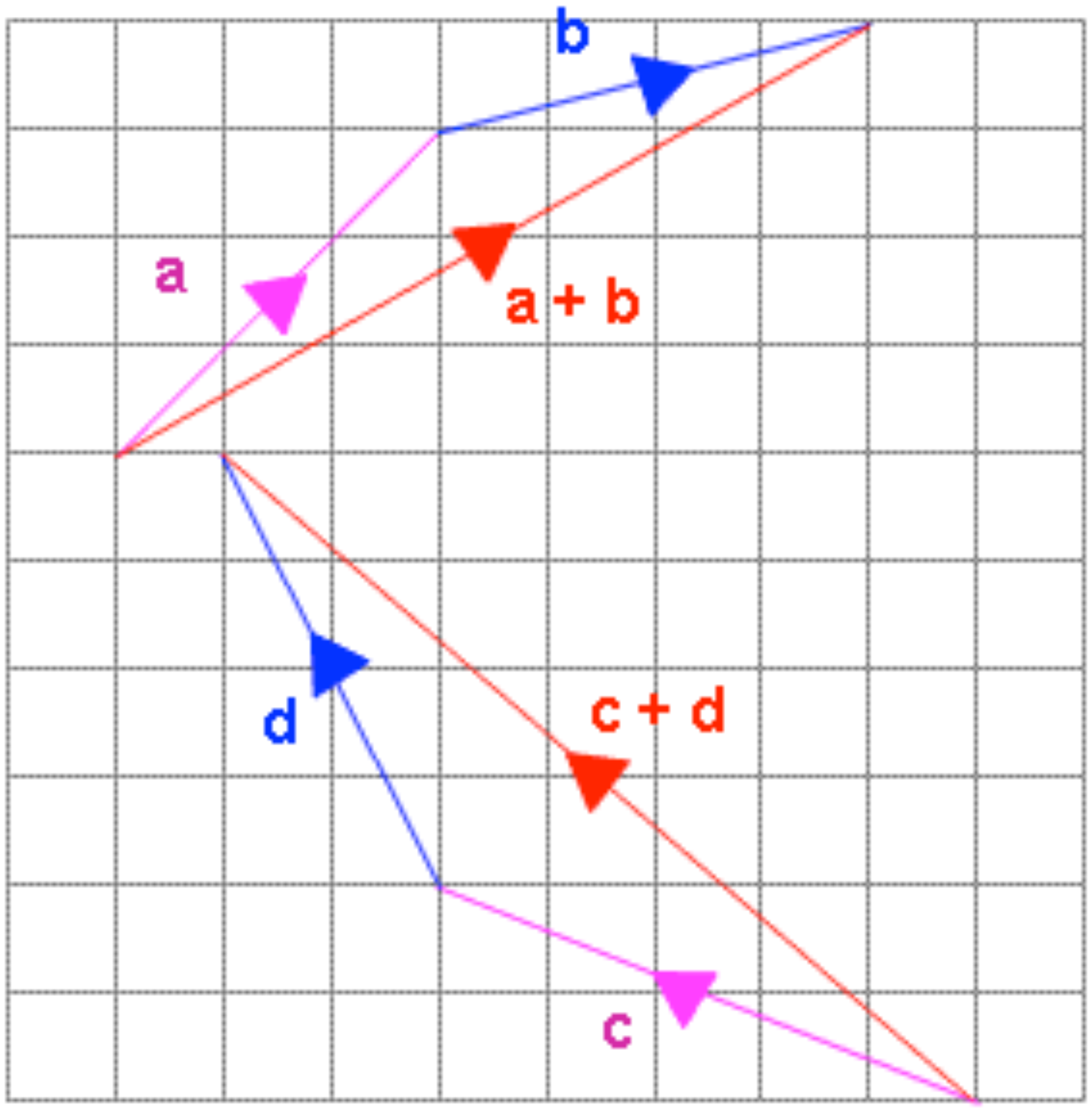
3. Adding Vectors

- When you add two or more vectors together, you simply add the tops and add the bottoms
- The new vector you end up with is called the resultant vector



a $\begin{pmatrix}3\\3\end{pmatrix}$ **b** $\begin{pmatrix}4\\1\end{pmatrix}$

a + b $\begin{pmatrix}3\\3\end{pmatrix}+\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}$

c $\begin{pmatrix}-5\\2\end{pmatrix}$ **d** $\begin{pmatrix}-2\\4\end{pmatrix}$

c + d $\begin{pmatrix}-5\\2\end{pmatrix}+\begin{pmatrix}-2\\4\end{pmatrix}=\begin{pmatrix}-7\\6\end{pmatrix}$

Watch Out! Remember to be careful with your negatives!

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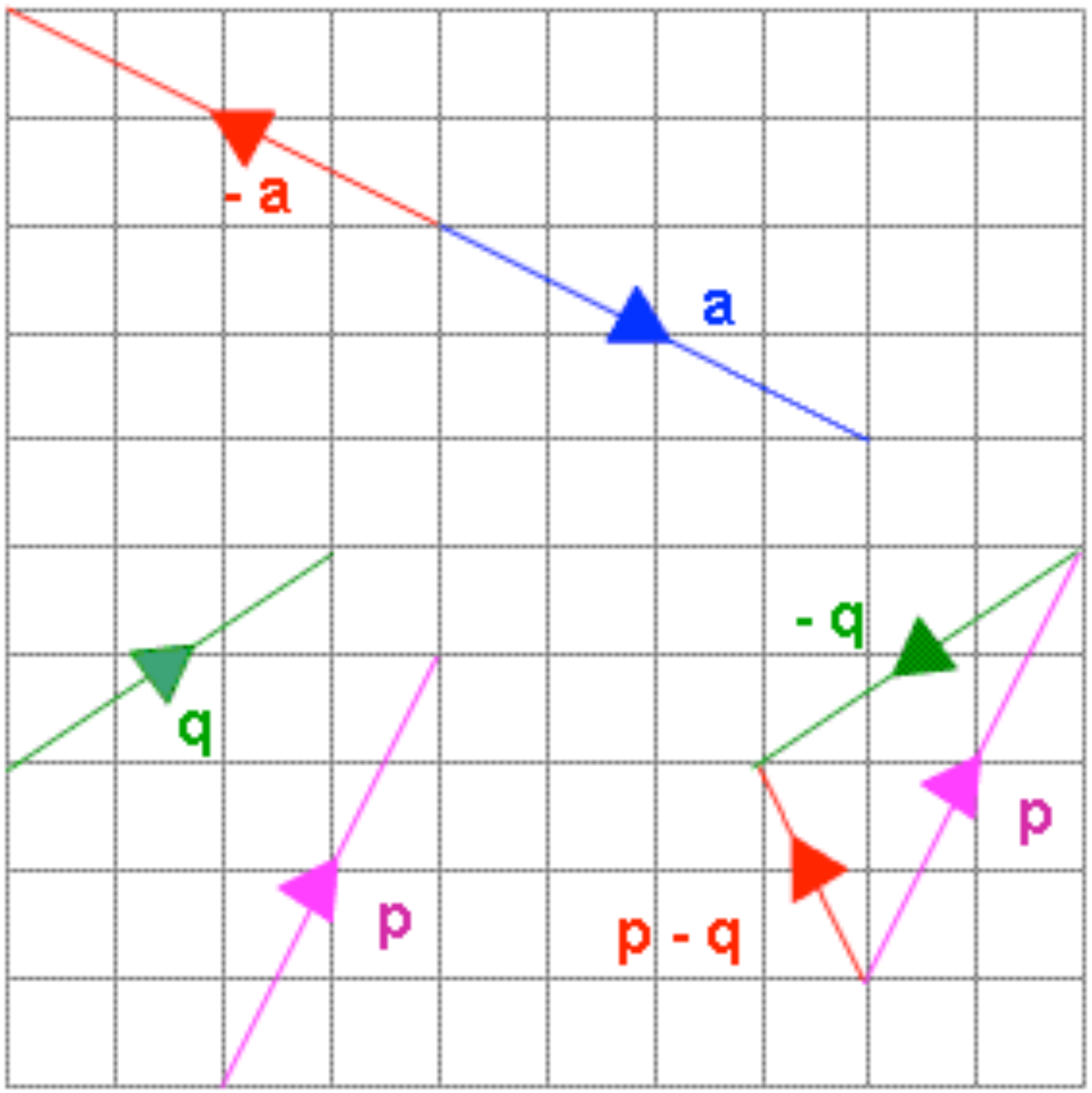
4. Subtracting Vectors

- The negative of a vector goes in the exact opposite direction, which changes the signs of the numbers on the top and the bottom (see below)
- One way to think about subtracting vectors is to simply add the negative of the vector!



$$\mathbf{a} \begin{pmatrix} 4 \\ -2 \end{pmatrix} \qquad -\mathbf{a} \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

$$\mathbf{p} \begin{pmatrix} 2 \\ 4 \end{pmatrix} \qquad \mathbf{q} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q})$$

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

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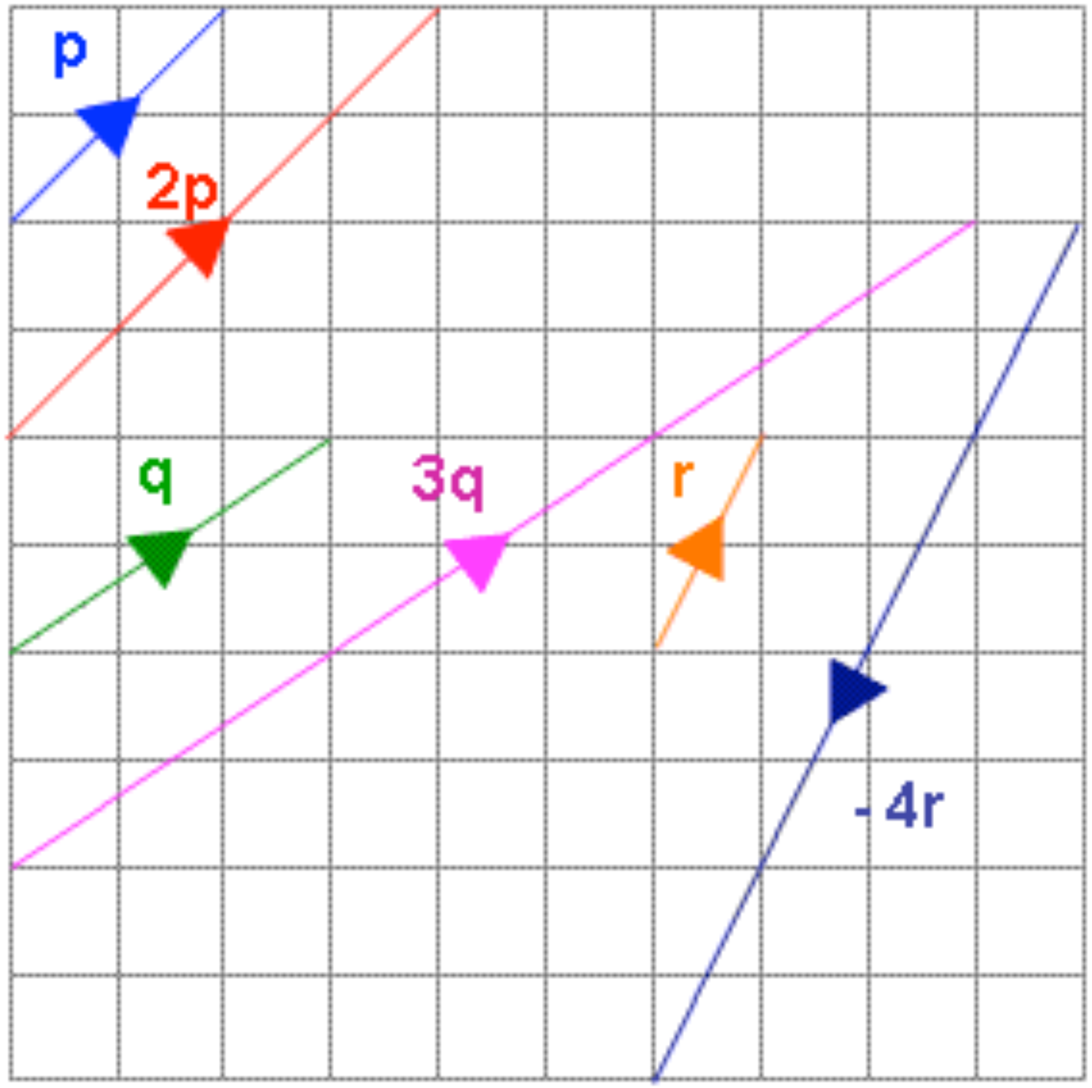
5. Multiplying Vectors

- The only thing you need to remember when multiplying vectors is that you multiply both the top and the bottom of the vector!



p $\begin{pmatrix}2\\2\end{pmatrix}$ 2p $2\begin{pmatrix}2\\2\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}$

q $\begin{pmatrix}3\\2\end{pmatrix}$ 3q $3\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}9\\6\end{pmatrix}$

r $\begin{pmatrix}1\\2\end{pmatrix}$ -4r $-4\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-4\\-8\end{pmatrix}$

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6. Linear Combinations of Vectors

- Using the skills we learnt when multiplying vectors, it is possible to calculate some pretty complicated looking combinations of vectors

Example: If $a = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ $b = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ $c = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ Calculate the following:

(a) 4a + 3b + c

$$4\begin{pmatrix} 3 \\ 5 \end{pmatrix} + 3\begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 20 \end{pmatrix} + \begin{pmatrix} -12 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 24 \end{pmatrix}$$

(a) 2a - 5b - 2c

$$2\begin{pmatrix} 3 \\ 5 \end{pmatrix} - 5\begin{pmatrix} -4 \\ 2 \end{pmatrix} - 2\begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 10 \end{pmatrix} - \begin{pmatrix} -20 \\ 10 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 28 \\ 4 \end{pmatrix}$$

Watch Out! Remember to be so, so careful with your negatives!

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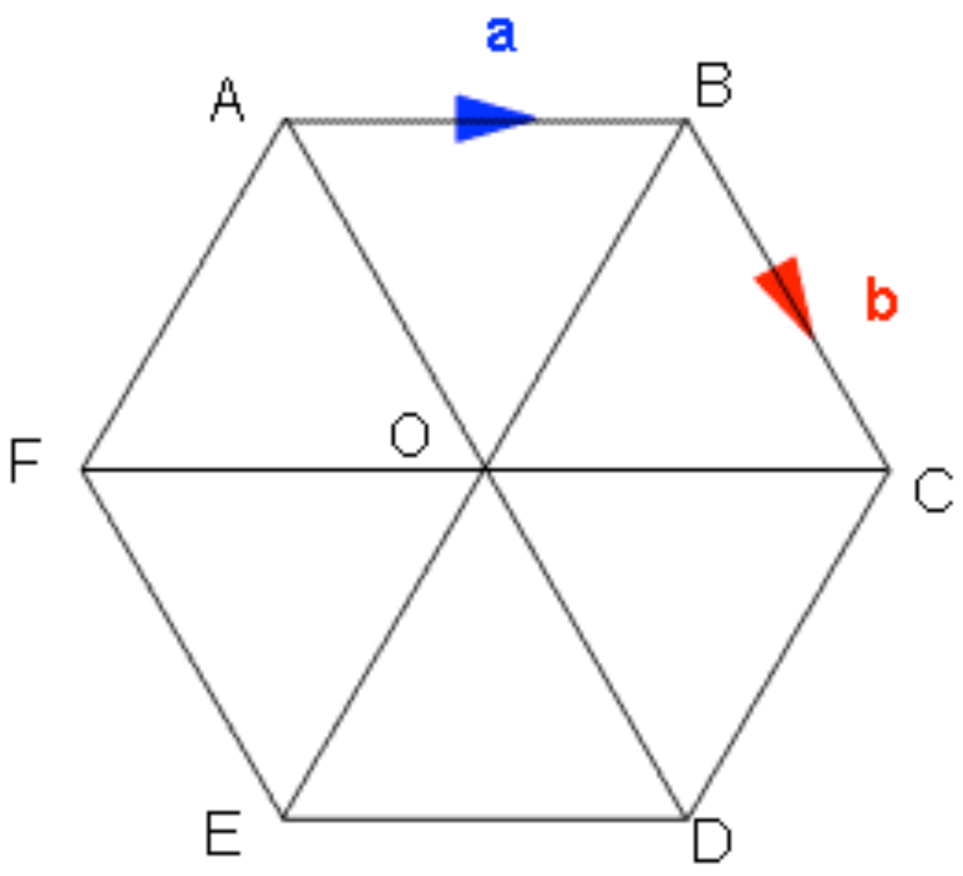
7. Vectors in Geometry

- A popular question asked by the lovely examiners is to give you a shape and ask you to describe a route between two points using vectors.
- There is one absolutely crucial rule here... you can only travel along a route of known vectors!

Just because a line looks like it should be a certain vector, doesn't mean it is!

Example: Below is a regular hexagon. Describe the routes given in terms of vectors **a** and **b**



(i) $\overrightarrow{FC}$

The best way to go here is straight across the middle, because we know each horizontal line is just **a**

$$\overrightarrow{FC} = 2\mathbf{a}$$

(ii) $\overrightarrow{DA}$

Again, the middle is looking good here, but remember we are going the opposite way to our given vector, so we need the negative!

$$\overrightarrow{DA} = -2\mathbf{b}$$

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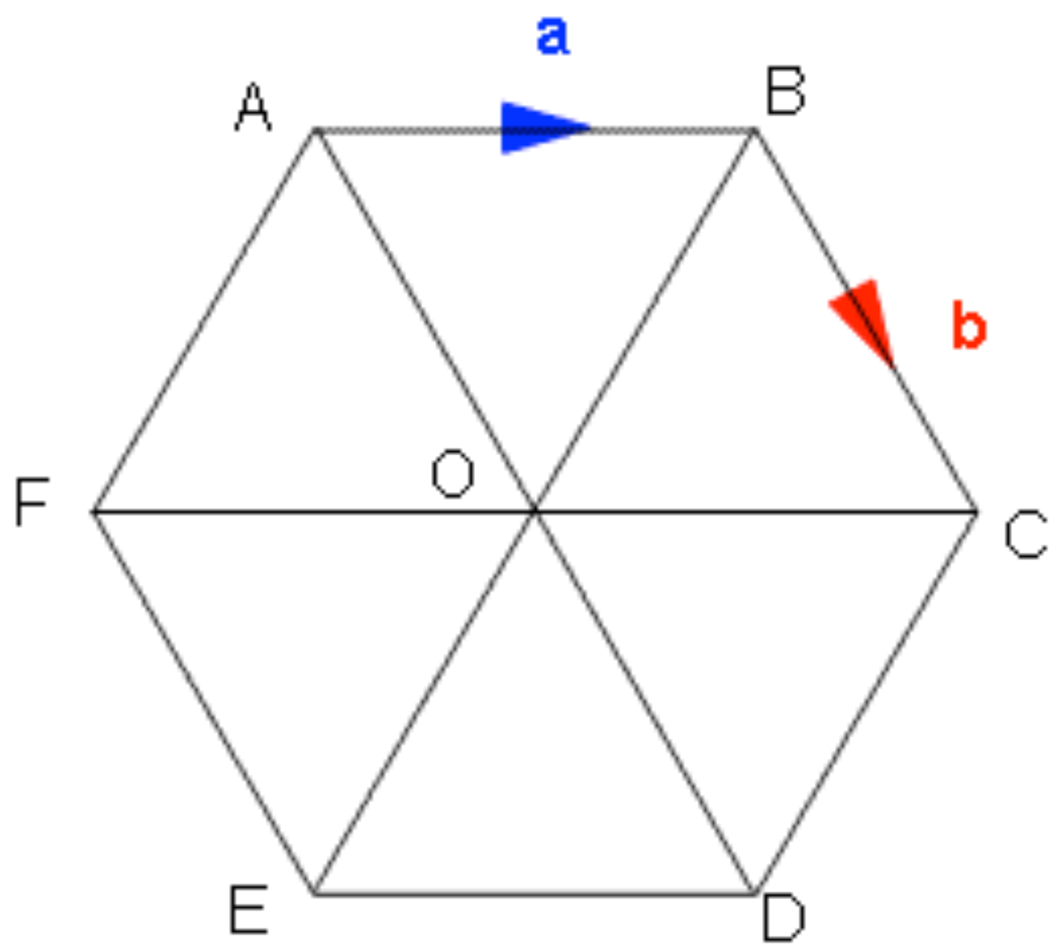

(iii) $\overrightarrow{EB}$

It would be nice to just nip across the middle, but the problem is we do not know what those vectors are! So… we'll just have to go the long way around, travelling along routes we do know!

$$\overrightarrow{EB} = \overrightarrow{EF} + \overrightarrow{FO} + \overrightarrow{OA} + \overrightarrow{AB}$$

$$= -b + a + -b + a$$

$$= 2a - 2b$$

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10. Transformations

What are Transformations and What do you need to be able to do?

- Transformations are specific ways of moving objects, usually around a co-ordinate grid

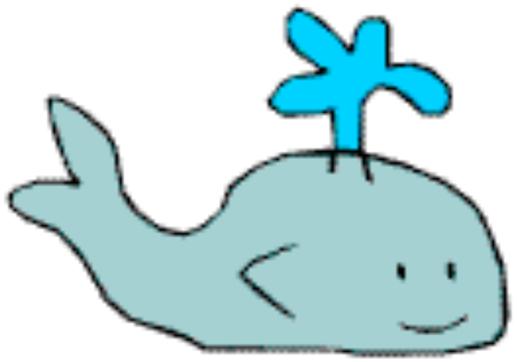
- There are 4 types of transformations you need to be clued up on, and for each one you must
  - Be able to carry out a transformation yourself
  - Be able to describe a transformation giving all the required information

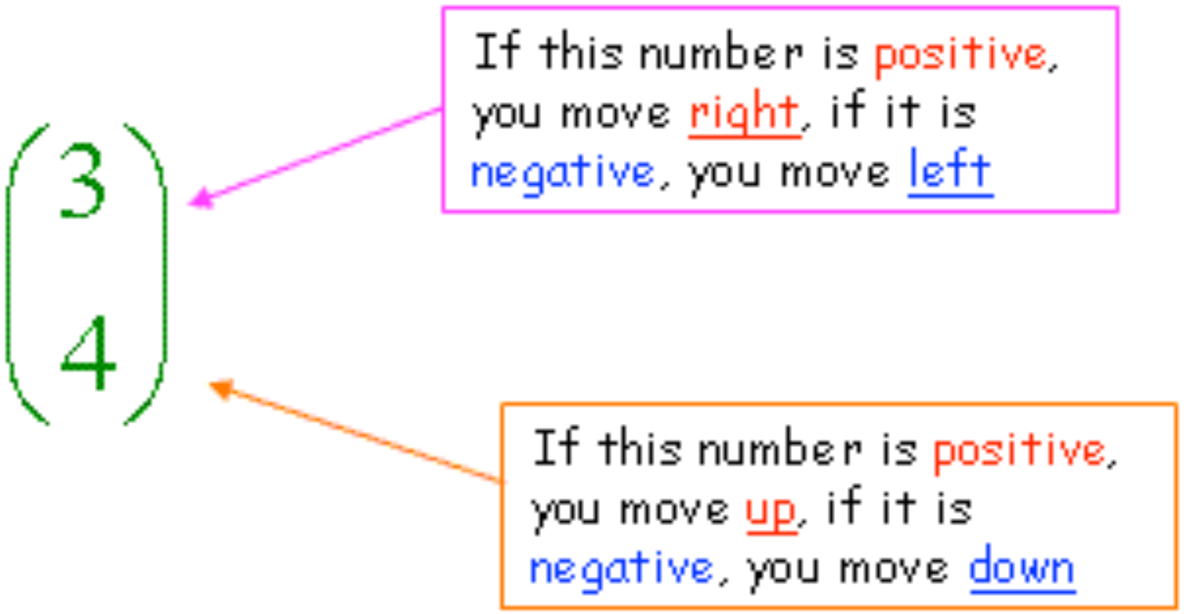
1. Translation

A Translation is a movement in a straight line, described by a movement right/left, followed by a movement up/down

Describing Translations

You must give the vector which describes the translation





$$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

If this number is positive, you move right, if it is negative, you move left

If this number is positive, you move up, if it is negative, you move down

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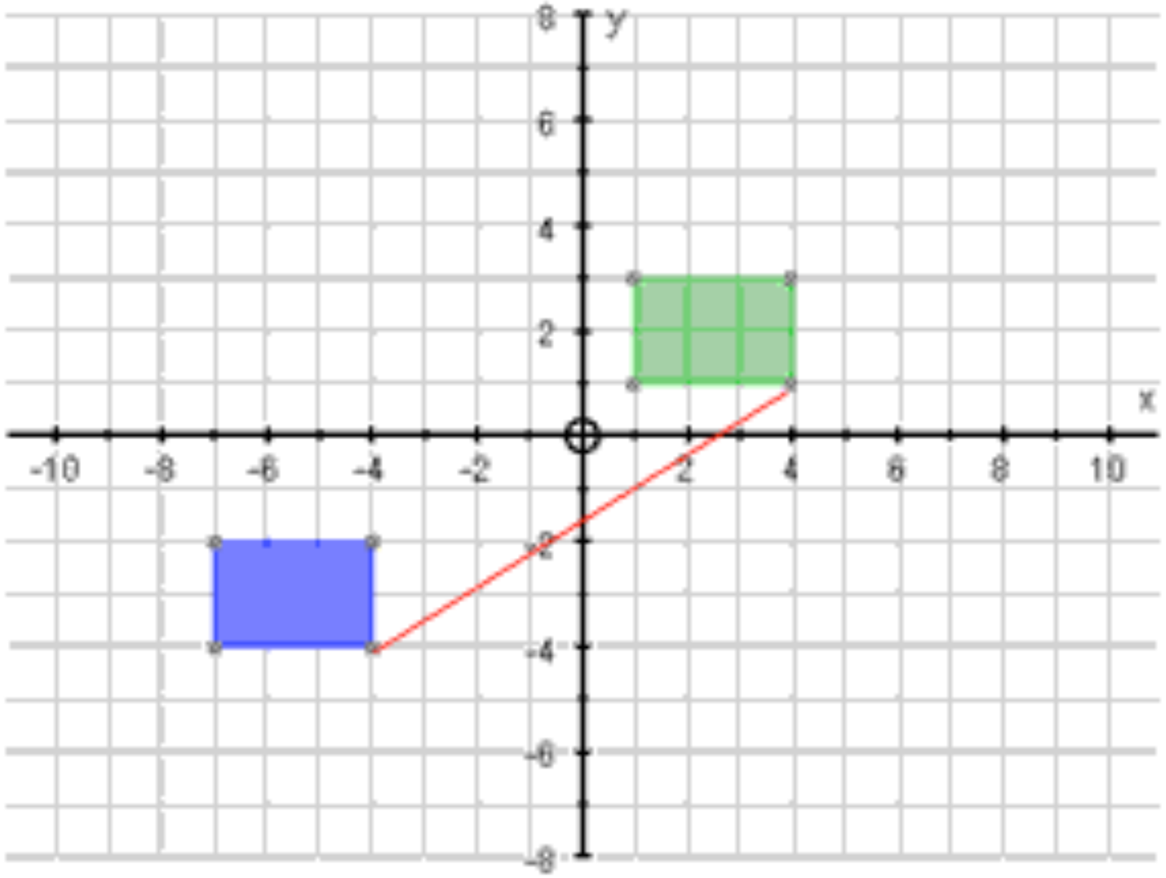


If we translate the blue object by the vector:

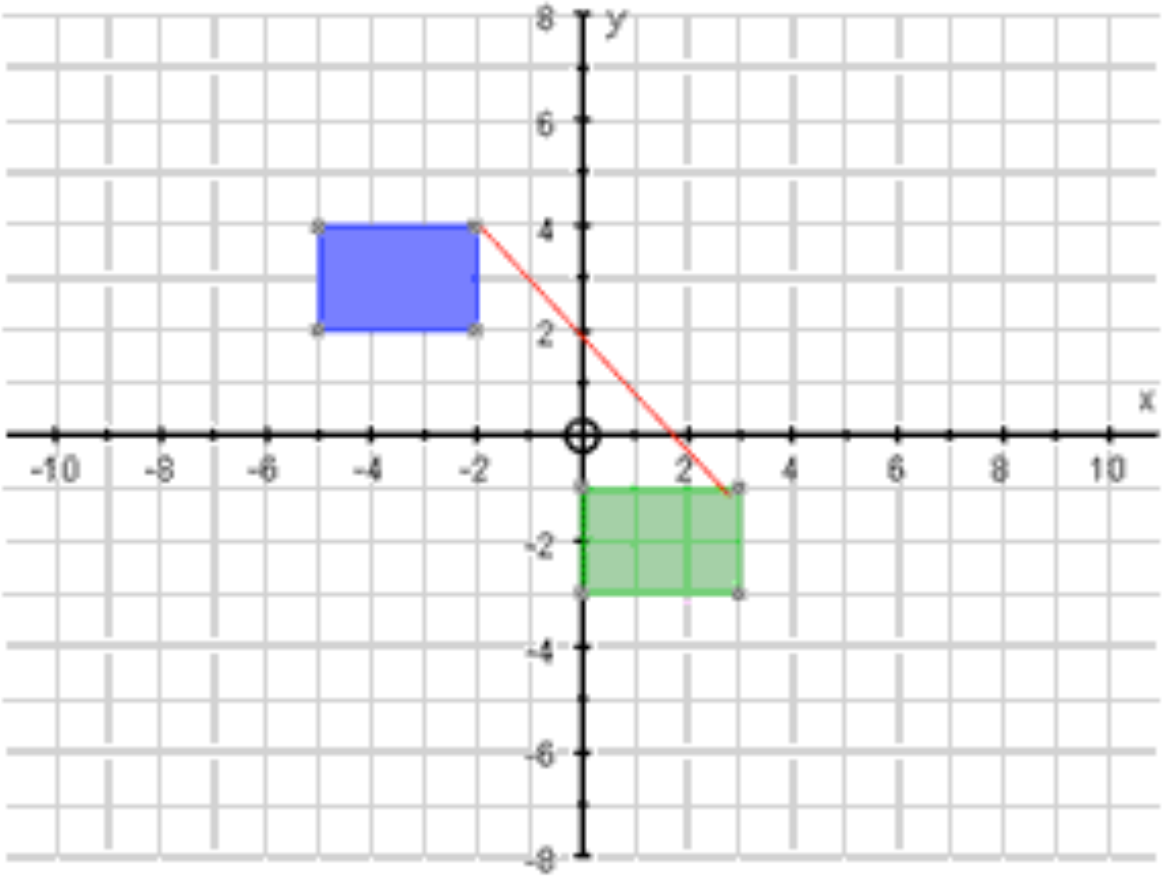
$$\begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

8 to the right

5 up

We end up with the green object

Notice: If you pick any co-ordinate on the blue shape and translate it by the same vector, you end up with the matching corner on the green shape



If we translate the blue object by the vector:

$$\begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

5 to the right

5 down

We end up with the green object

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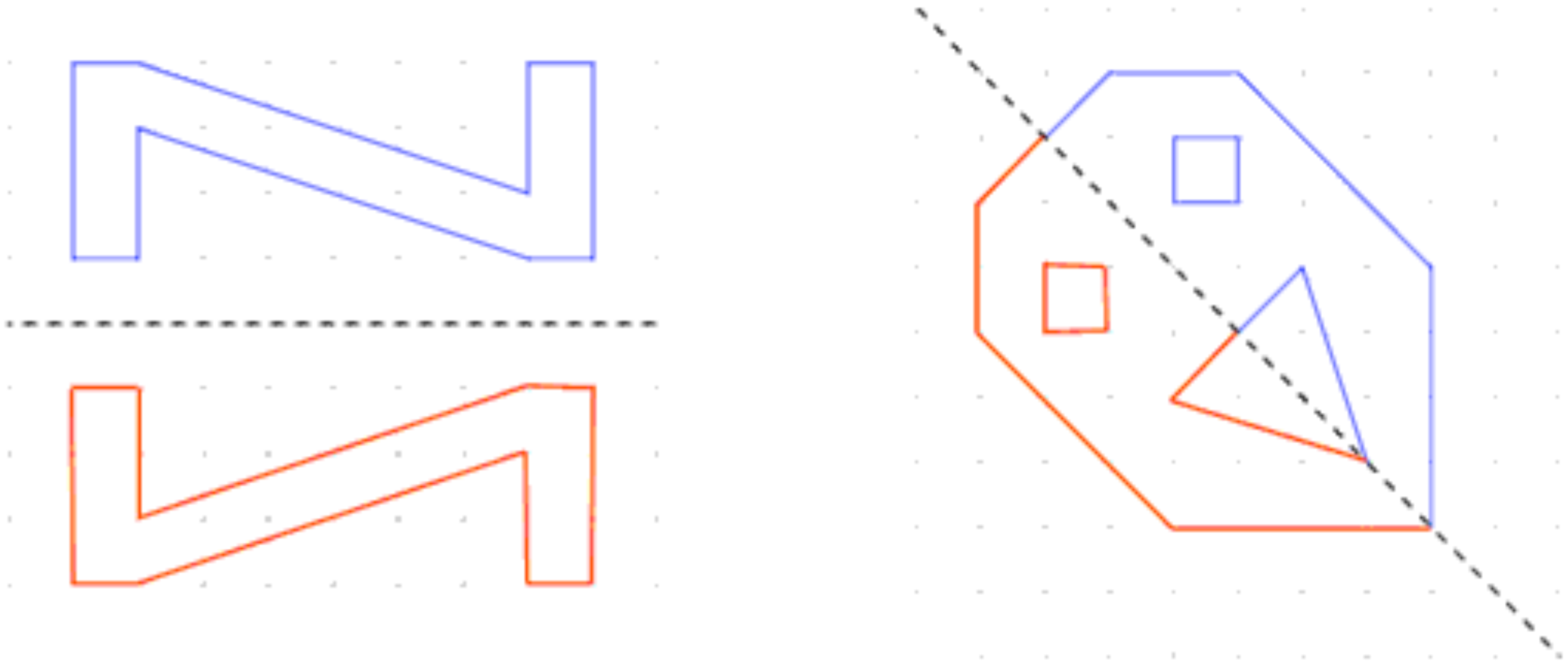
2. Reflection

Reflecting an object across a line produces an exact replica (mirror image) of that object on the other side of the line.
This new shape is called the Image



Describing Reflections

You must give either the equation of the line of reflection (mirror line) or draw the line on the grid.

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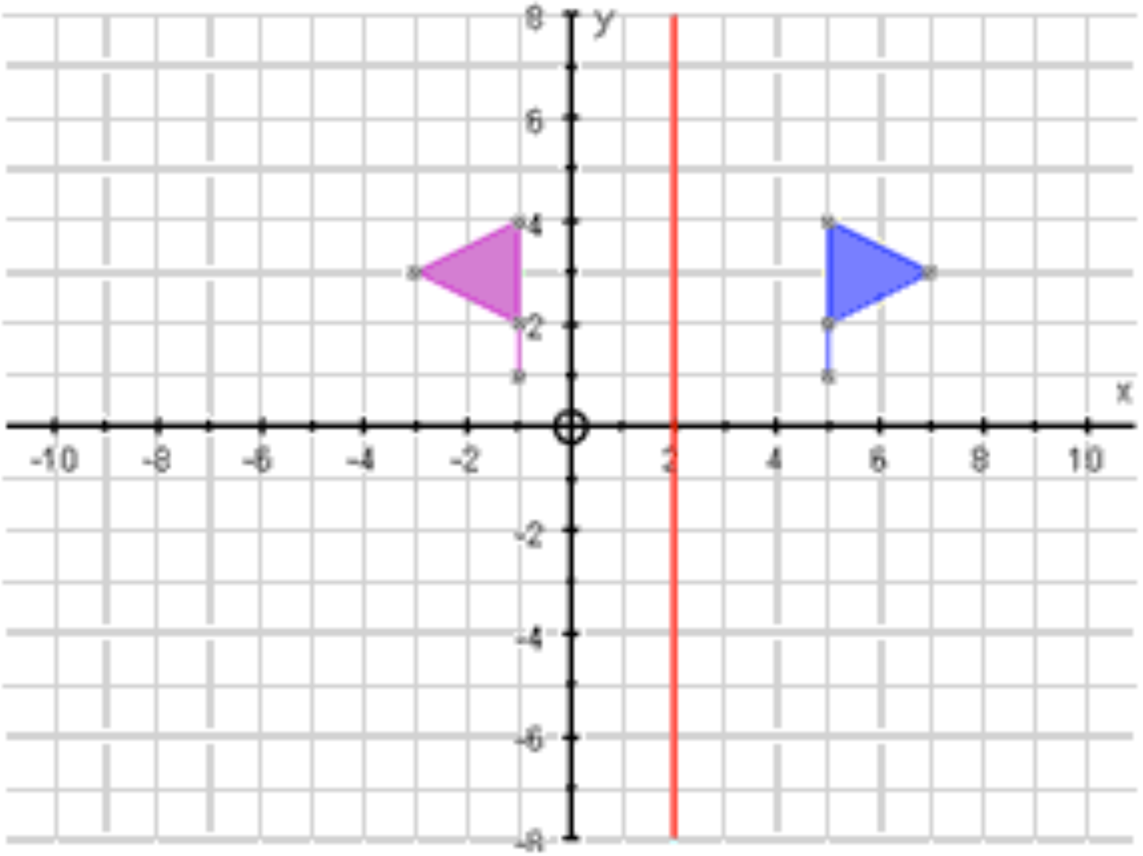
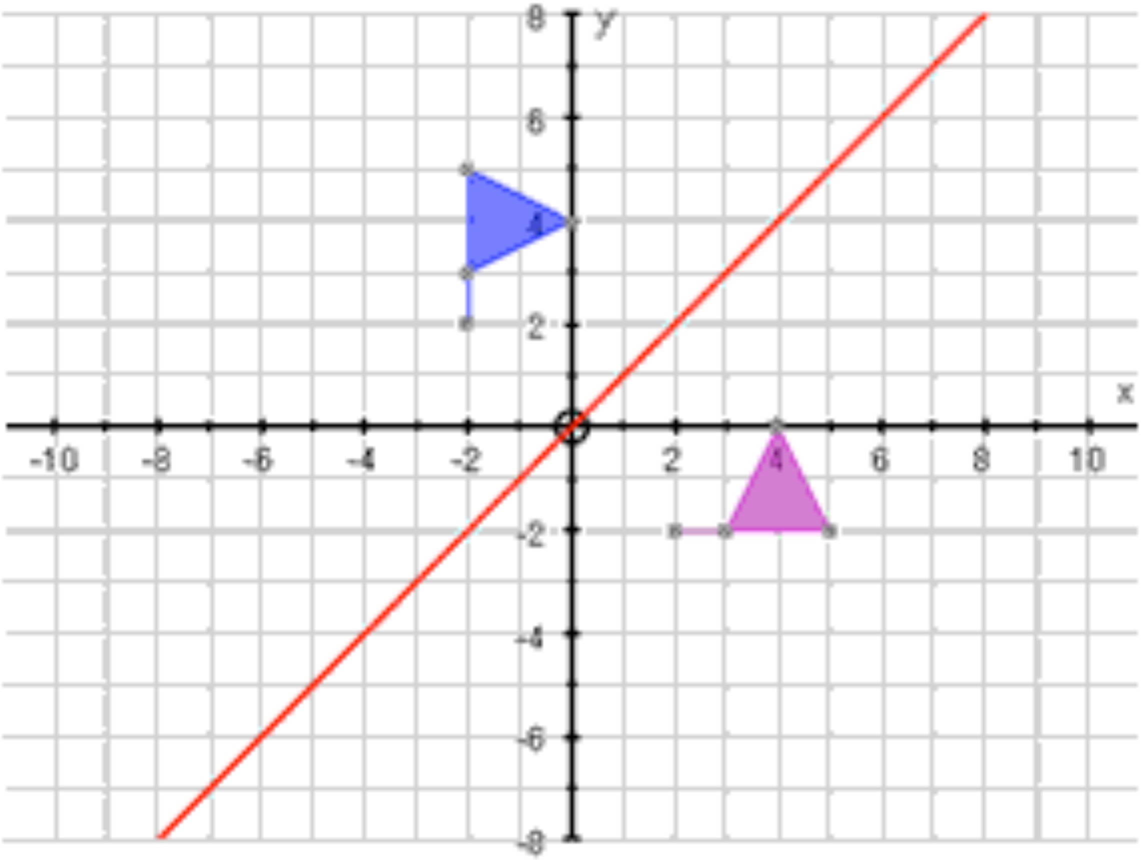


If we reflect the blue object in the red line (equation: $x = 2$), we end up with the purple object

Notice: Every point on the purple object (the image) is the exact same distance from the line of reflection as the matching point on the blue object



If we reflect the blue object in the red line (equation: $y = x$), we end up with the purple object

Notice: I find it much harder to reflect when the mirror line is diagonal, but notice how every point on the image is still the same distance away from the mirror line as the matching point on the original object.

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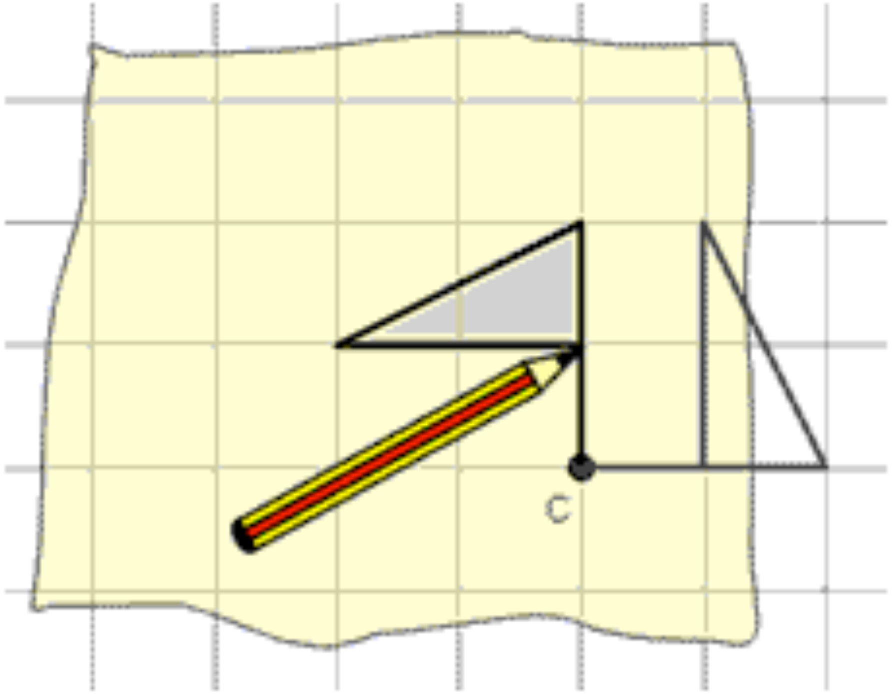
3. Rotation

Rotating an object simply means turning the whole shape around a fixed point by a certain number of degrees and in a certain direction!

Remember: If you a like Mr Barton and you can't do these in your head, then all you need to do is:

- trace around the object
- place your pencil at the centre of rotation (the fixed point)
- turn the tracing paper around
- draw your rotated object!



Describing Rotations

Warning: People always forget to give all the information here and lose loads of easy marks!

You must give all of the following:

1. The centre of rotation (give as a co-ordinate if you can)

2. The direction of the rotation (clockwise or anti-clockwise)

3. The angle of the rotation (usually either 90^0, 180^0 or 270^0)

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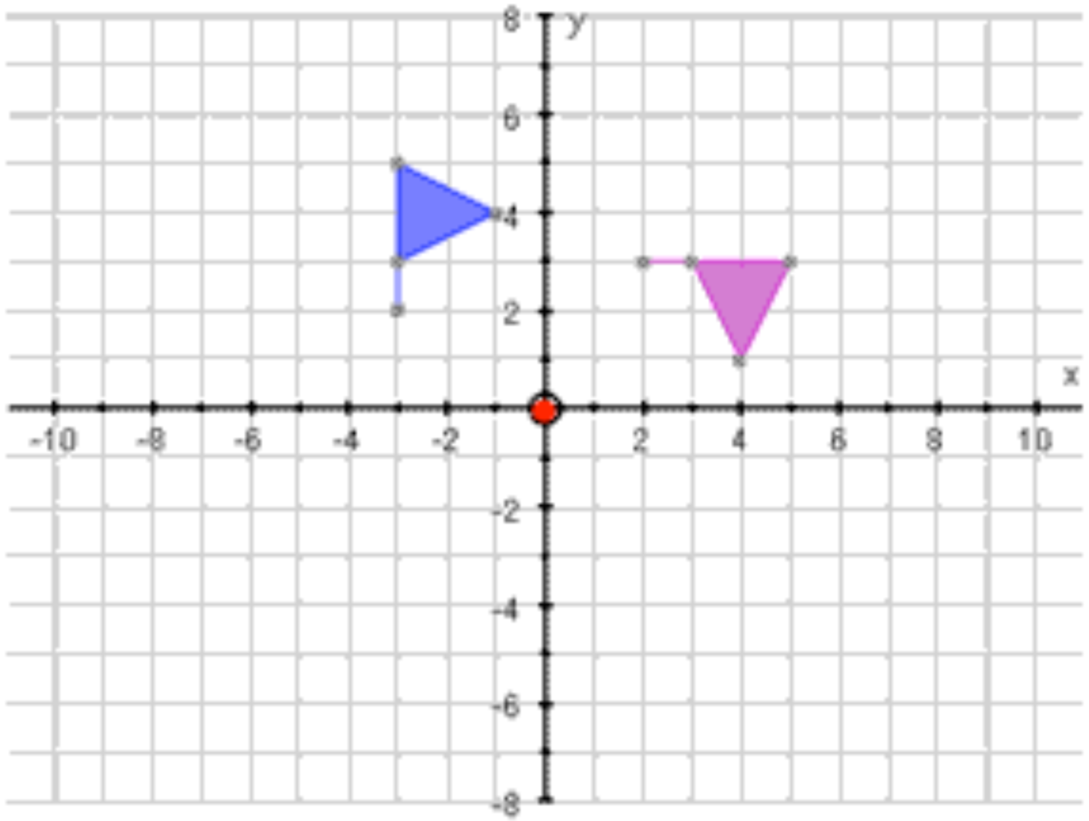


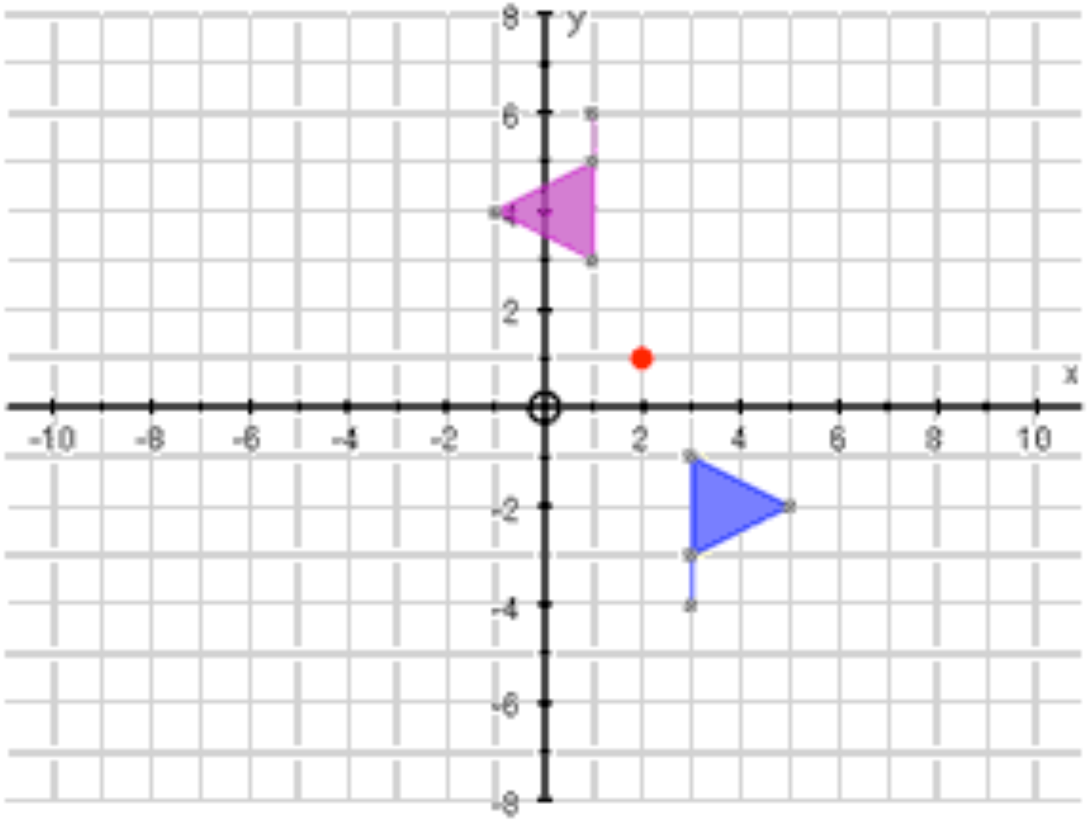
To describe the rotation from the blue object to the purple object, we would say:

1. Centre of Rotation: (0, 0) – the origin

2. Direction of Rotation: Clockwise

3. Angle of Rotation: 90°

Notice: If you wanted to be clever, you could also say it was an anti-clockwise 270° rotation!



To describe the rotation from the blue object to the purple object, we would say:

1. Centre of Rotation: (2, 1)

2. Direction of Rotation: Clockwise

3. Angle of Rotation: 180°

Notice: Whenever the angle of rotation is 180°, it doesn't matter whether you go clockwise or anti-clockwise!

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4. Enlargement

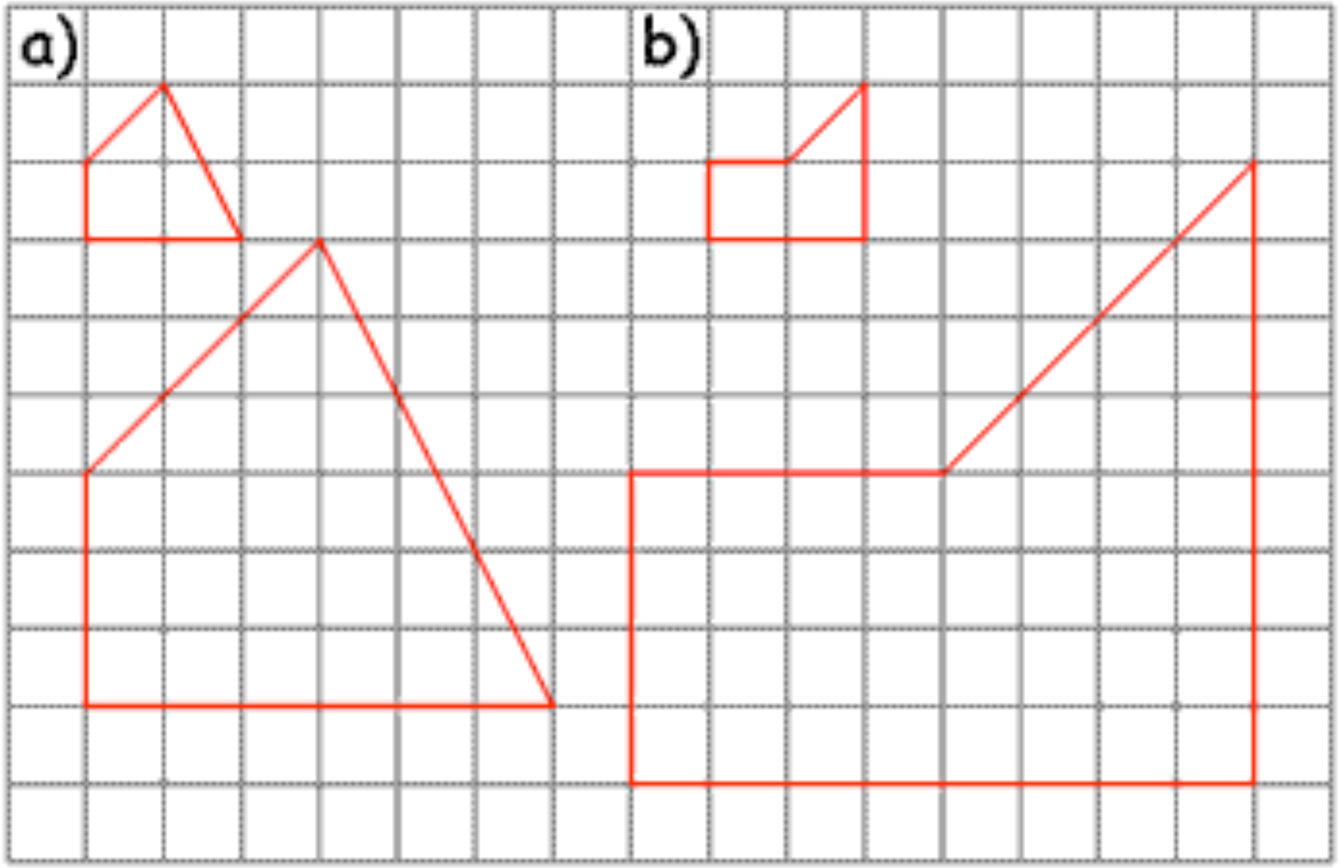
Enlargement is the only one of the four transformations which changes the size of the object

Key Point: Enlargements can make objects bigger as well as smaller!

Each length is increased or decreased by the same scale factor

(a) Scale Factor = 3
(b) Scale Factor = 4

And going from big to small...
(a) Scale Factor = $\frac{1}{3}$
(b) Scale Factor = $\frac{1}{4}$



Describing Enlargements

To fully describe an enlargement, you must give:

1. The centre of enlargement (give as a co-ordinate if you can)

2. The scale factor of the enlargement

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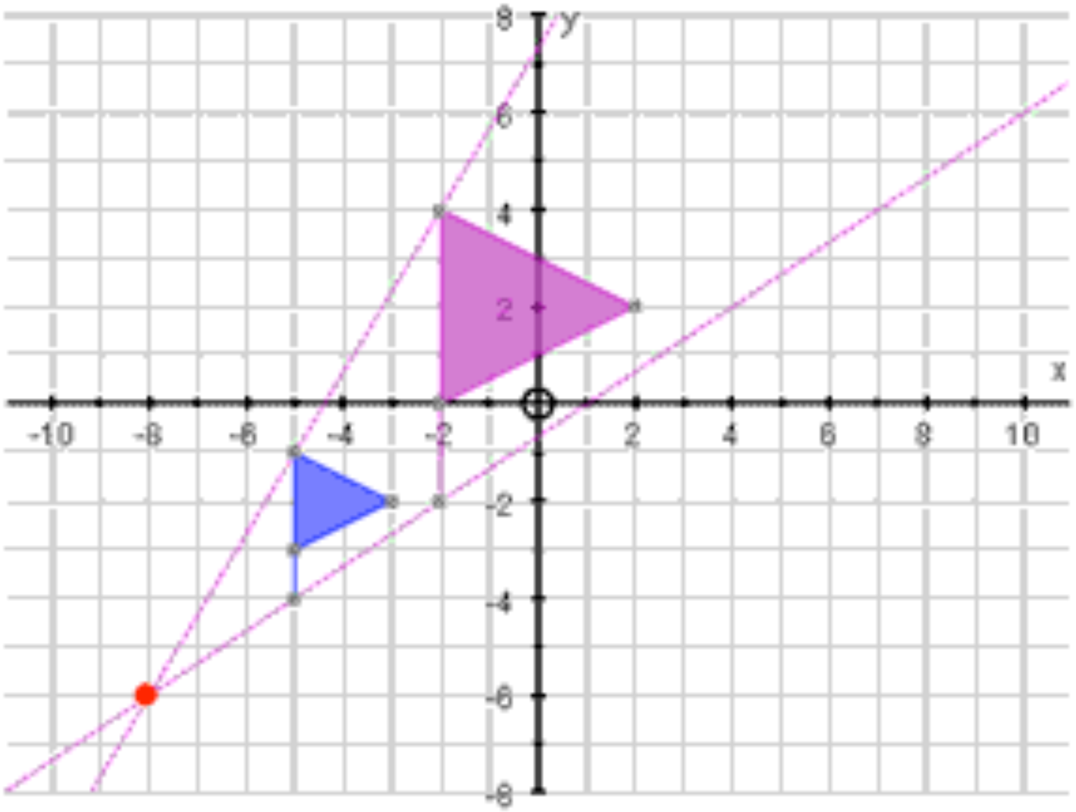


To describe the enlargement from the blue object to the purple object, we would say:

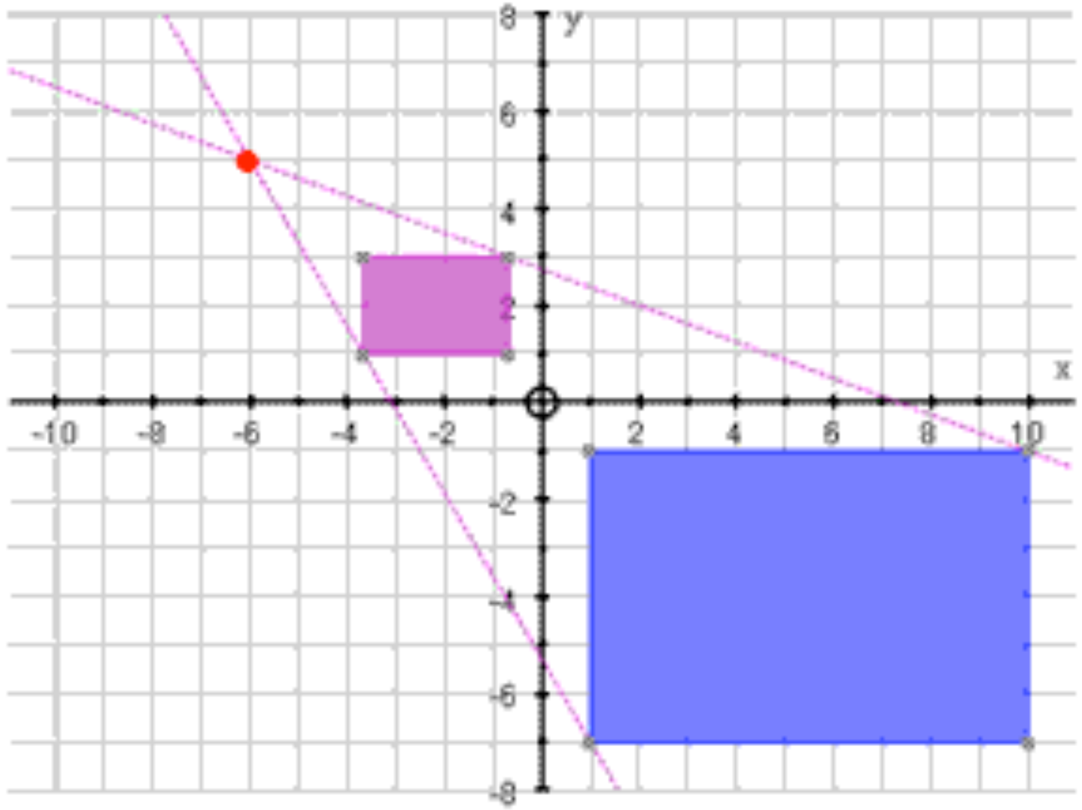
1. Centre of Enlargement: (-8, -6)

2. Scale Factor of Enlargement: 2

Notice:

(1) To find the centre of enlargement you must draw line through matching points on both objects and see where they cross

(2) Each point on the purple object if twice as far away from the centre of enlargement than the matching point on the blue!



To describe the enlargement from the blue object to the purple object, we would say:

1. Centre of Enlargement: (-6, 5)

2. Scale Factor of Enlargement: $\frac{1}{3}$

Notice:

(1) The object has gone smaller, so it must be a fractional scale factor!

(2) Each point on the purple object if one-third as far away from the centre of enlargement than the matching point on the blue!

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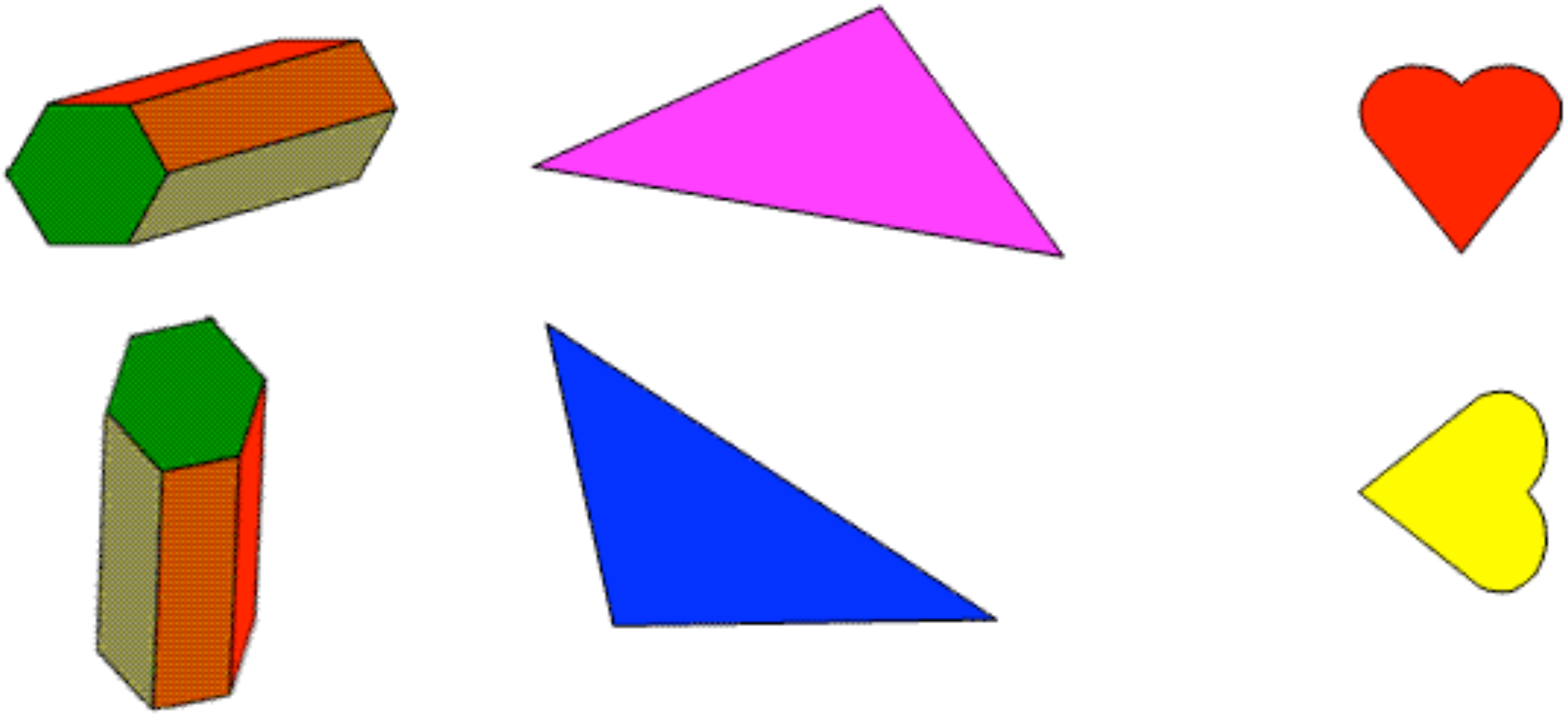
11. Similarity and Congruency

1. If two shapes are Congruent, what does that mean?

When mathematicians say that two shapes are congruent, it is just a posh, complicated way of saying that those shapes are IDENTTICAL

They may have been flipped upside down and rotated around, but they are still exactly the same shape and the same size



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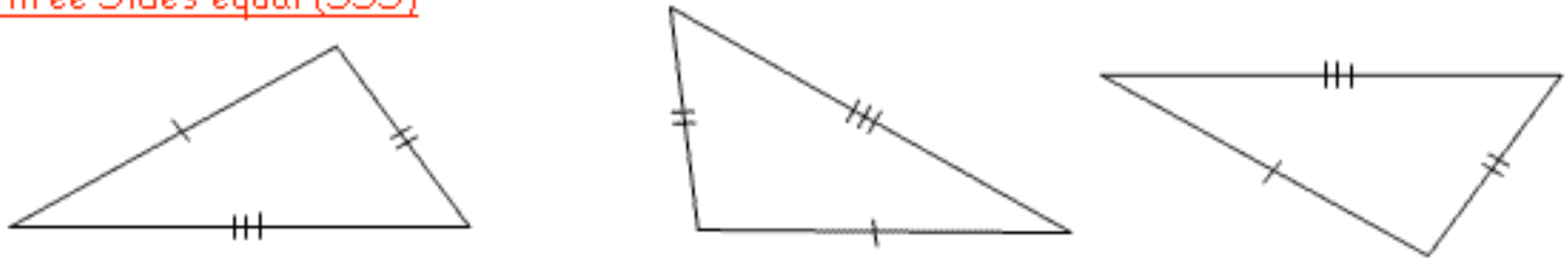
2. Congruent Triangles

Because triangles only have three sides, and we know that all their interior angles must add up to 180^0, we don't actually need to know every single piece of information about two triangles to be able to say that they are congruent (identical).

There are 4 sets of criteria, and if a pair of triangles match any of these, then we can say for definite that they are the exact same triangle, and so they are congruent!

1. Three Sides equal (SSS)



The lengths of all three sides are given in the question, and they are the same for both triangles

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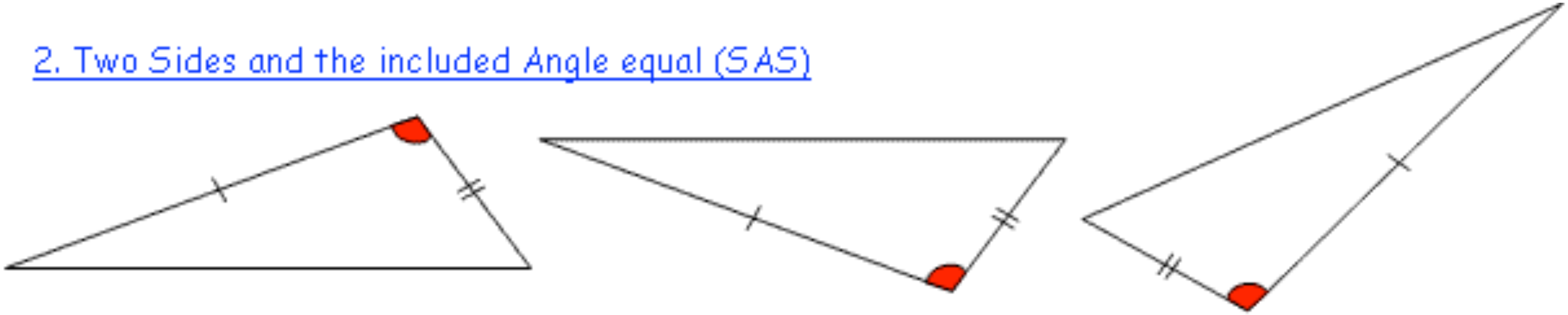
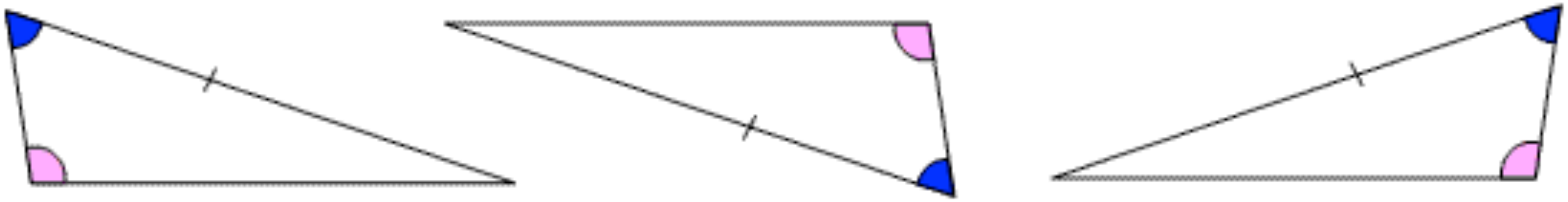
2. Two Sides and the included Angle equal (SAS)



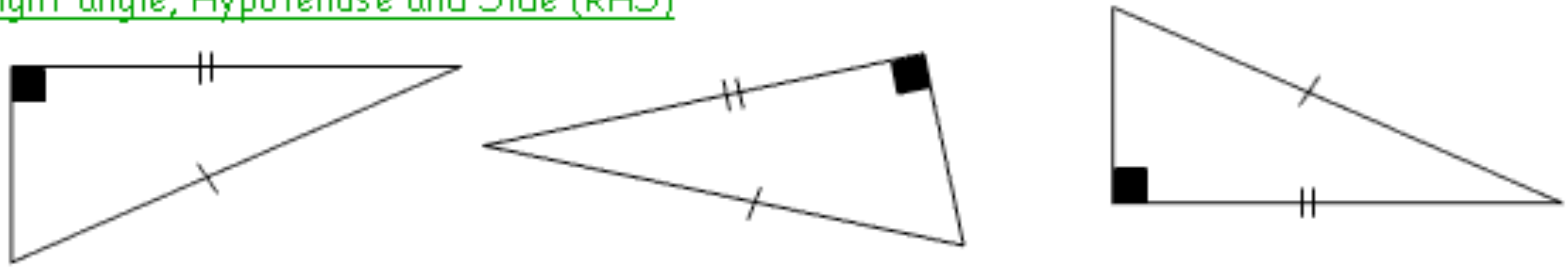
Two sides are the same length, and the angle in between those two sides is the same size!

3. Two Angles and a corresponding Side equal (AAS)



Two angles are equal, and so too is a side in the same position relative to those two angles!

4. Right angle, Hypotenuse and Side (RHS)



The triangle has a right angle, and you know the length of the hypotenuse and another side!

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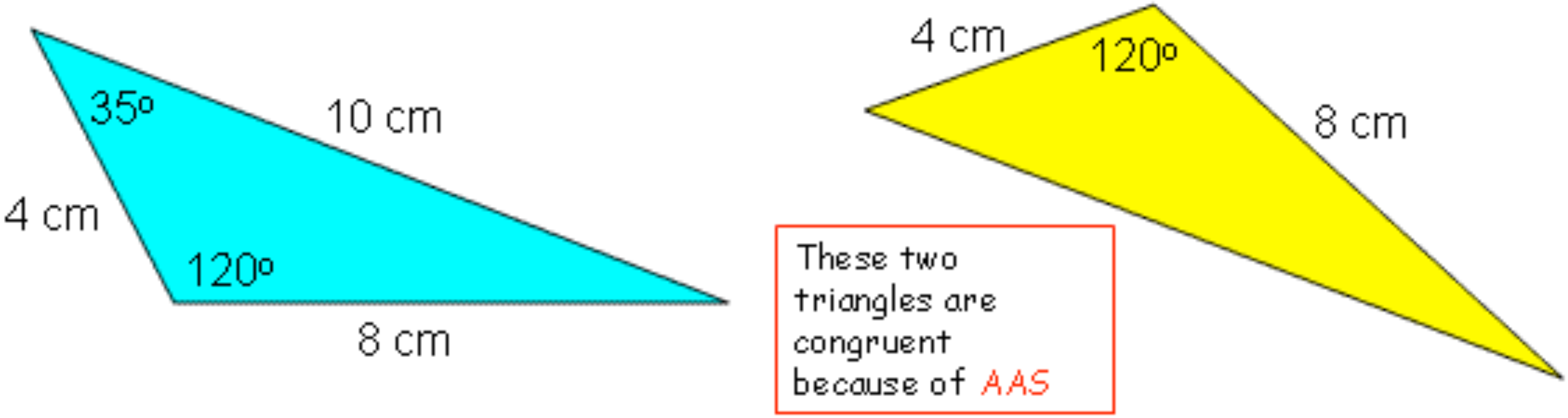
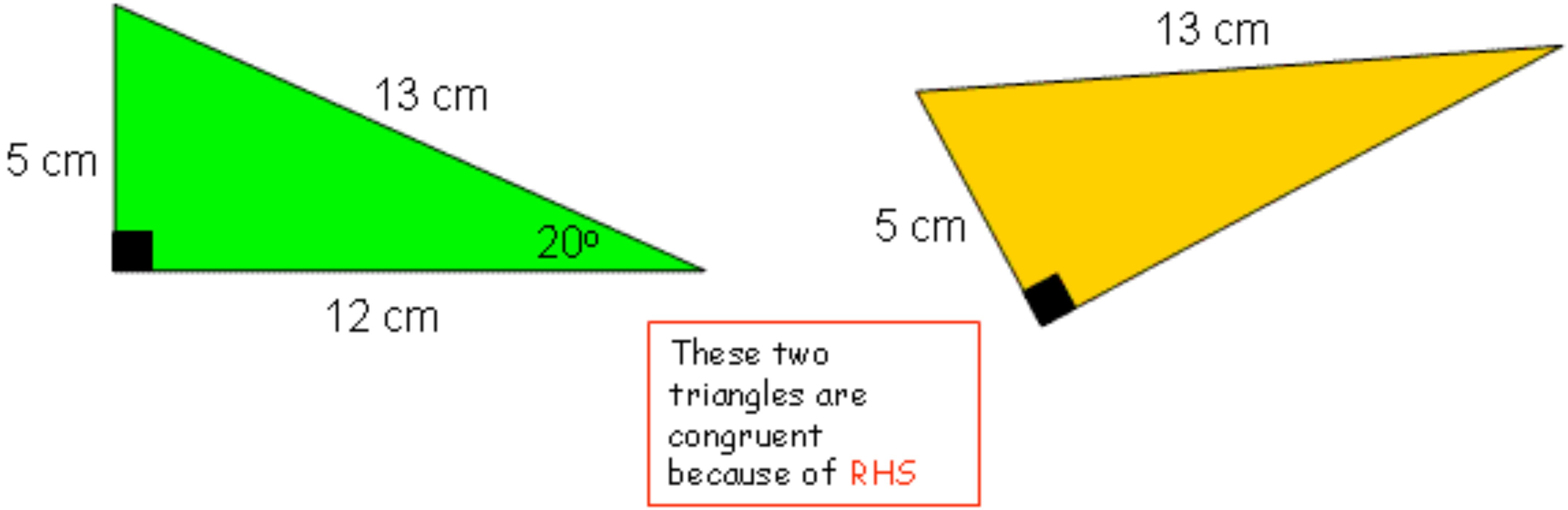
3. Examples

When answering questions on congruent triangles, you must quote one of the above four conditions if you believe a pair of triangles to be congruent:



These two triangles are congruent because of AAS



These two triangles are congruent because of RHS

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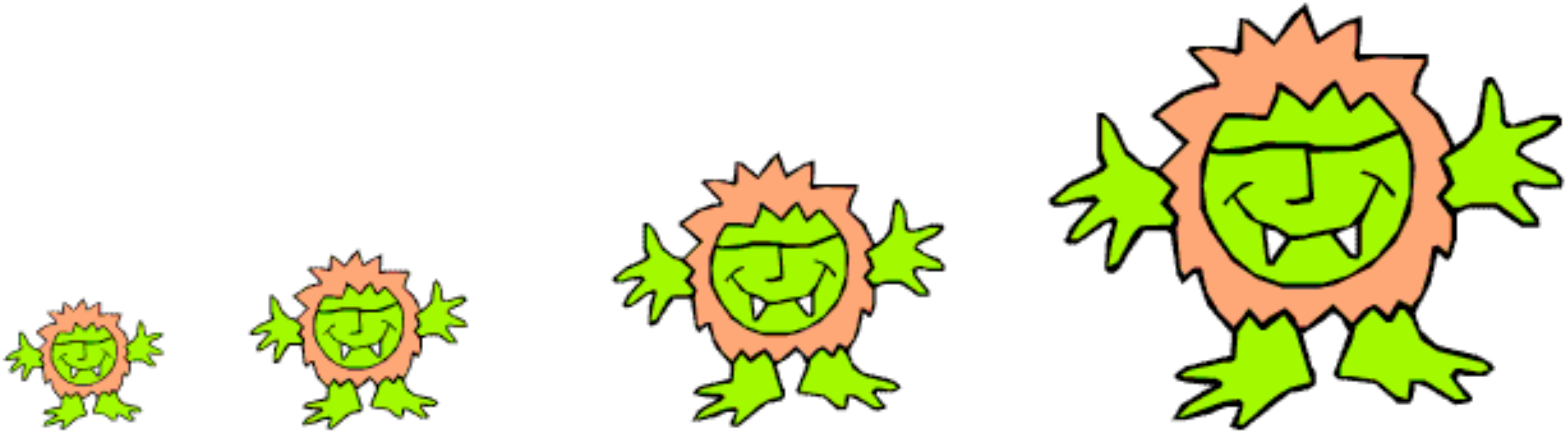
4. If two shapes are Similar, what does that mean?

- Unfortunately, when mathematicians says that two objects are similar, they do not mean that they look a bit a like

- They mean that one object is an enlargement of the other



- Technically, to get from one object to the other you must multiply (or divide) every single length by the same number

- Just like when we dealt with Enlargement, this number is called the Scale Factor!

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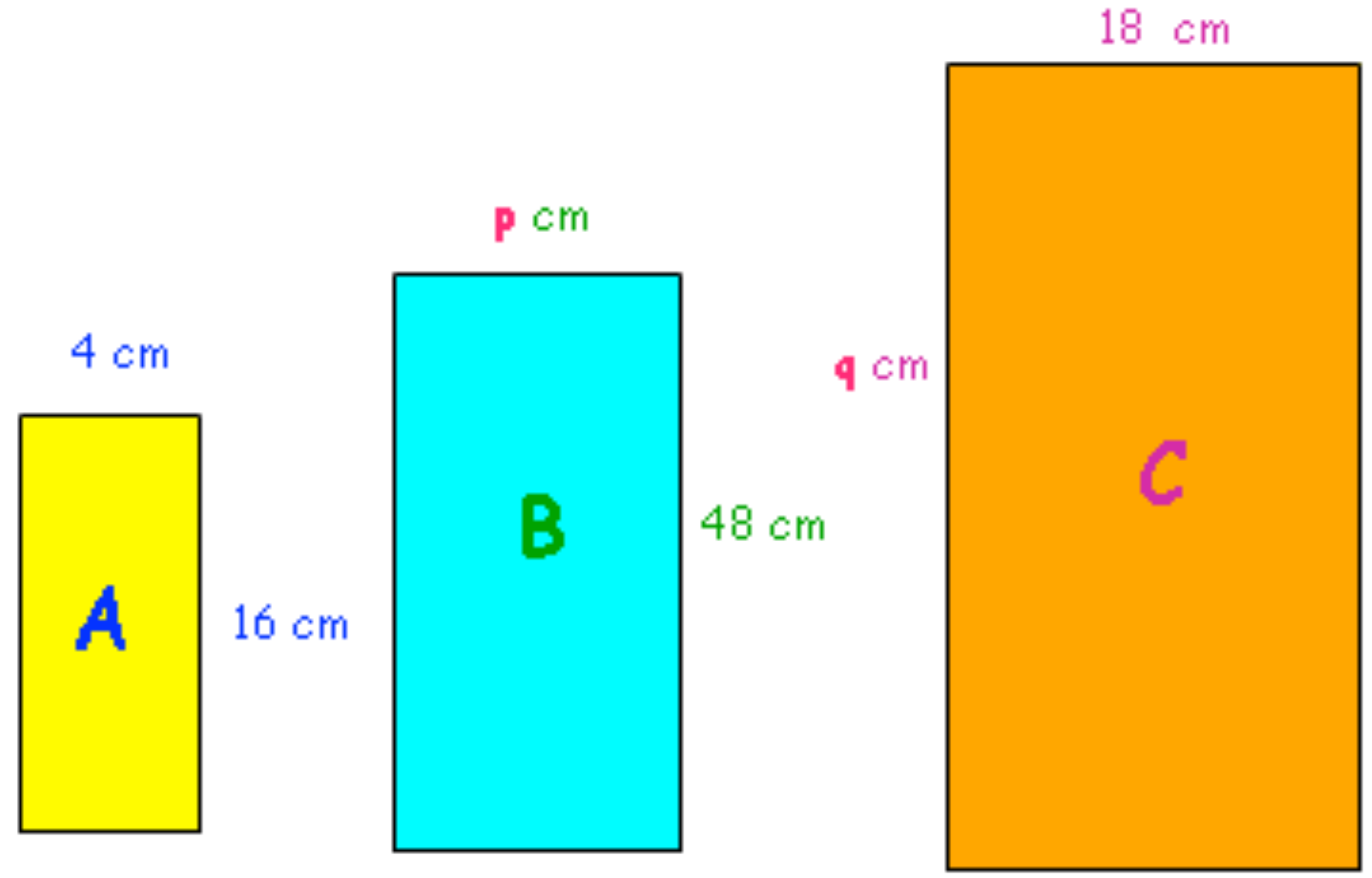
5. Using Length Scale Factors

If we are told that two object are similar, and we can work out the scale factor, then it is possible to work out a lot of unknown information about both objects

Example - These three shapes are similar. Find the missing values



To Find p:

Okay, so we know the shapes are similar, so let's work out the scale factor between rectangles A and B:

$$48 \div 16 = 3$$

So, we must enlarge every length on Rectangle A by a scale factor of 3 to get the lengths of Rectangle B.

So, our missing length must be:

$$4 \times 3 = 12\text{cm}$$

To Find q:

Okay, so now let's work out how to get from Rectangle A to Rectangle C

$$18 \div 4 = 4.5$$

So now we have our scale factor, it's dead easy to work out our missing length:

$$16 \times 4.5 = 72\text{cm}$$

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6. Similar Triangles

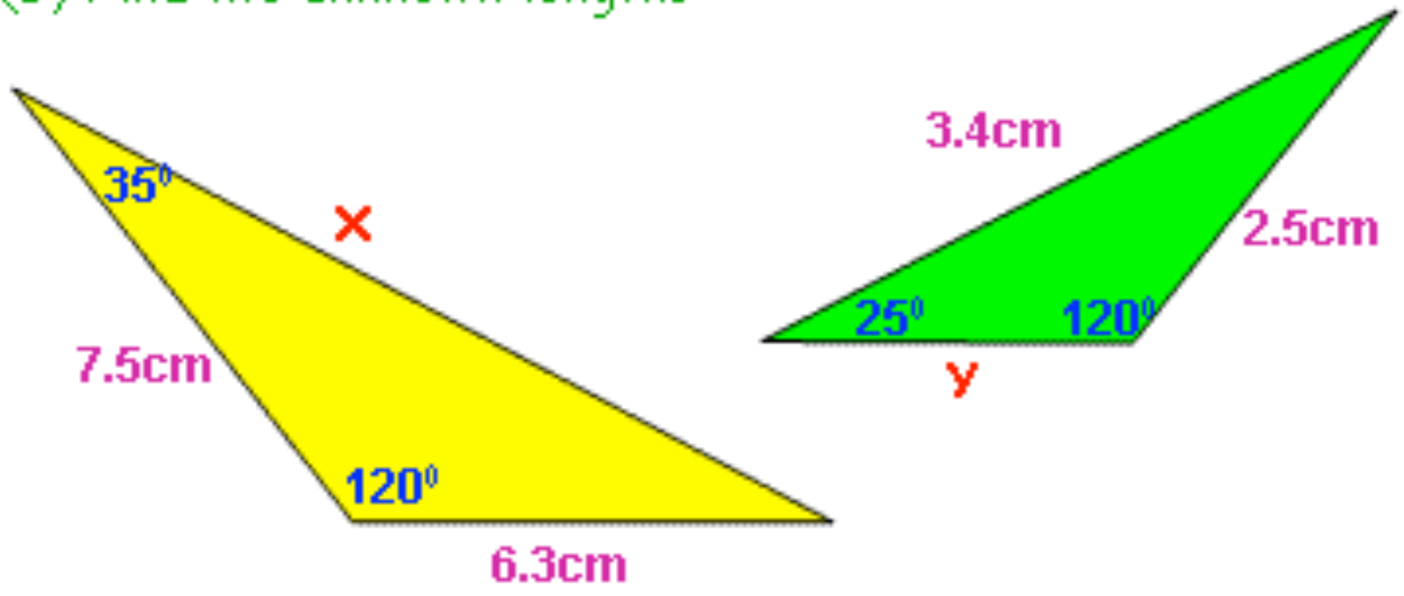
For any other shape to be similar, all angles must be the same and all matching sides must be in proportion

But... because triangles are funny, all you need for similarity between two triangles is for all three angles to be the same. Then you can be sure one triangle is an enlargement of the other

Example

(a) How do you know these two triangles are similar?

(b) Find the unknown lengths



To Find X

$$3.4 \times 3 = 10.2\text{cm}$$

To Find Y

$$6.3 \div 3 = 2.1\text{cm}$$

Part (a)

Two triangles are similar if all their angles are the same...

Well... if you work out the missing angle in the yellow triangle it is 25°, and the missing angle in the green triangle is... 35°

So... all the angles are the same, so the triangles are similar!

And because they are similar, we can work out the scale factor, using our matching sides between the 120° and the 35°...

$$7.5 \div 2.5 = 3$$

So, to get from one triangle to the other, we either multiply or divide by 3!

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7. Area and Volume Factors

It is also possible for 3D shapes to be similar.
If we can work out the scale factor between their lengths of sides, we can also say that:

Area Factor = Scale Factor^2
Volume Factor = Scale Factor^3

Example - These two containers are similar. Work out the volume of water the smaller one can hold

Okay, before we can do anything we need to work out the length scale factor in exactly the same way as we always do:

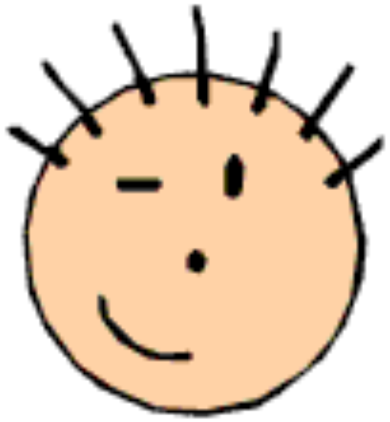
$$60 \div 40 = 1.5$$

So, if our length scale factor = 1.5

Volume Scale Factor = 1.5^3 = 3.375

So now we know how to get from the big container to the small container, so we can work out its volume:

$$20.25 \div 3.375 = 6 \text{ litres}$$



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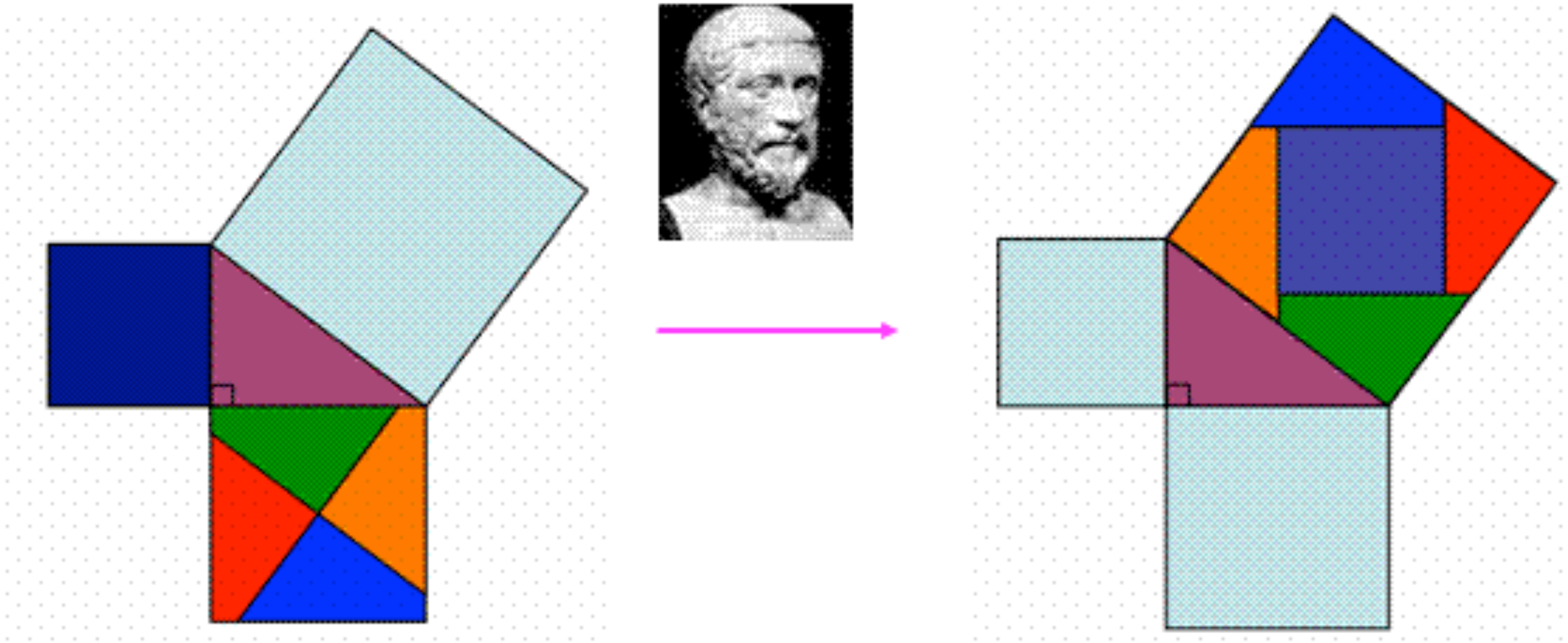
1. Pythagoras

What is Pythagoras' Theorem?

- Pythagoras' Theorem is probably the most famous theorem in the history of mathematics
- It was "invented" by a Greek named Pythagoras (or one of his loyal followers who always marked any of their discoveries with the Pythagoras brand) somewhere around 6BC
- Pythagoras discovered a very important relationship between the lengths of sides in a right-angled triangles:

"If you take the lengths of the two shortest sides of any right-angled triangle, square them and add the answers together, you end up with the square of the longest side (the hypotenuse)"



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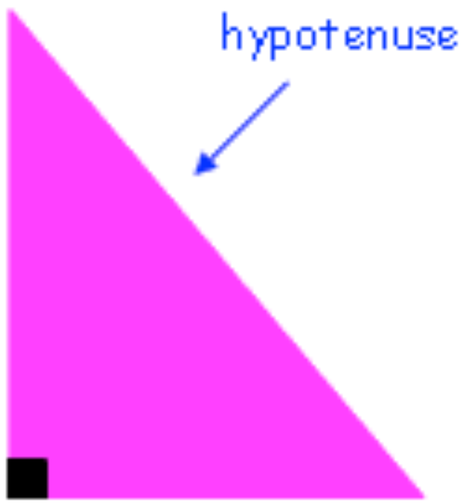
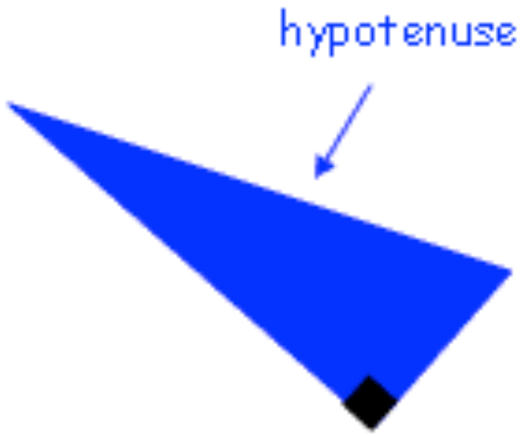
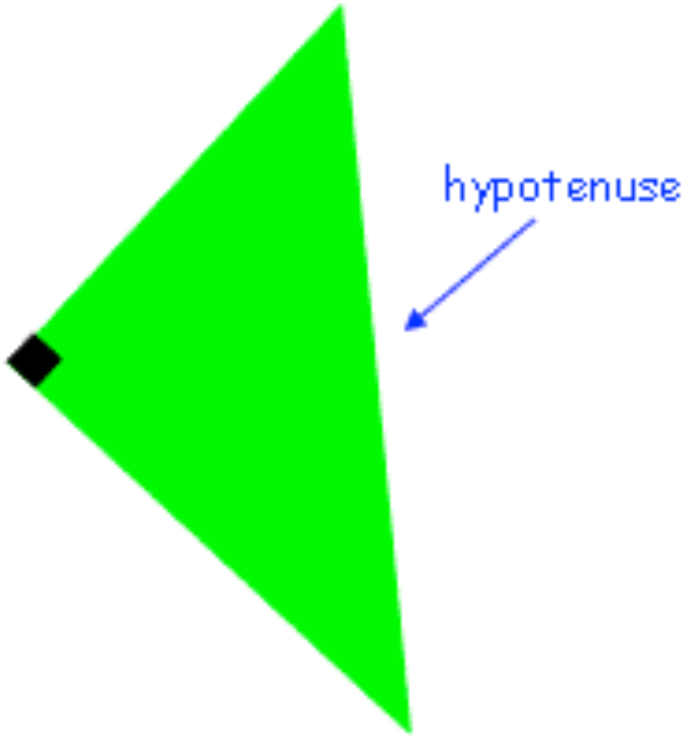
2. What is the Hypotenuse?

- In order to use Pythagoras' Theorem (or all the trig that is coming around the corner!), you must be an expert at finding the Hypotenuse of any right angled triangle

- The Hypotenuse is the longest side of the right-angled triangle, and it is the side opposite the right-angle!







3. The two forms of Pythagoras' Theorem

- Pythagoras' Theorem can be written in two ways depending on whether you want to find the length of the hypotenuse of a triangle, or one of the other sides.

- The two ways are just different arrangements of the same original formula, so if you are good at formula re-arranging, then you only need to remember one!

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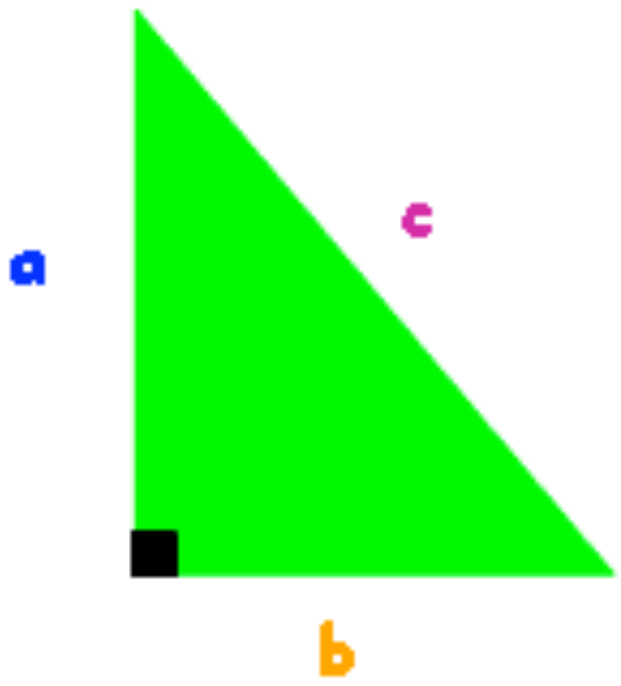
4. Finding the Hypotenuse

1. Label the Hypotenuse **c**, and the other sides **a** and **b**

2. Use the following formulae:



$$c^2 = a^2 + b^2$$

3. Replace the letters with the numbers you have been given, and carefully do the sum!

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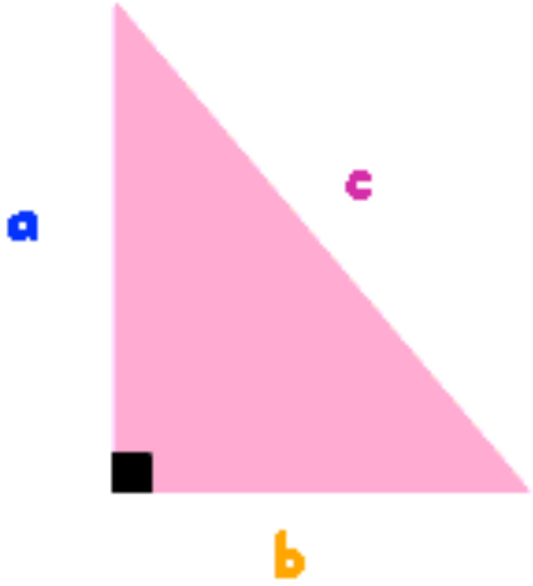
5. Finding a side that isn't the Hypotenuse

1. Label the Hypotenuse **c**, label the side you want to find **a**, and the other side **b**

2. Use the following formulae:



$$a^2 = c^2 - b^2$$

3. Replace the letters with the numbers you have been given, and carefully do the sum!

Note: As I mentioned before, this version of the formula is just a different arrangement of:

$$c^2 = a^2 + b^2$$

Just subtract b^2 from both sides and you should see what I mean!

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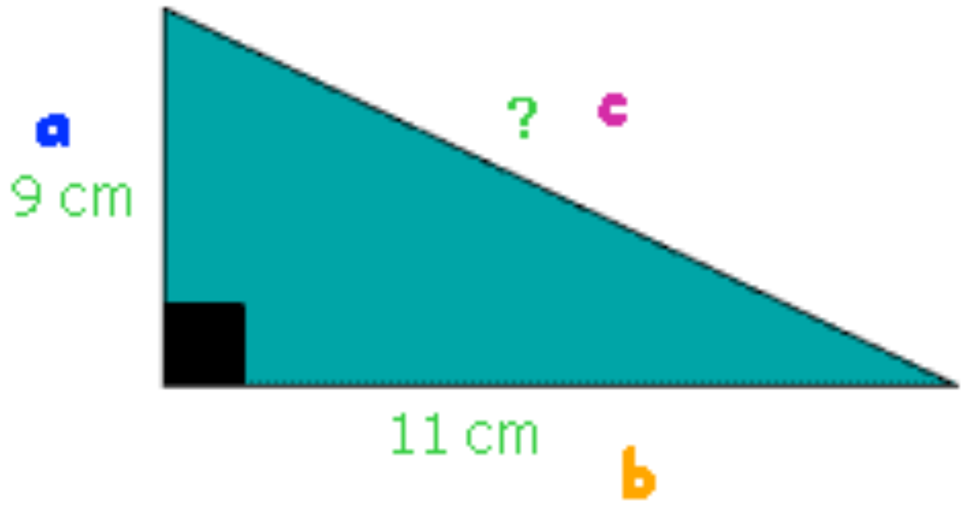
Examples

1.



Okay, so the side we want to find is the Hypotenuse, so let's go through our routine:

1. Label the sides

2. Use the formula: $c^2 = a^2 + b^2$

3. Put in the numbers:

$$c^2 = 9^2 + 11^2$$

$$c^2 = 81 + 121$$

$$c^2 = 202$$

Square root both sides!

$$c = \sqrt{202}$$

$$c = 14.2cm \quad (1\text{dp})$$

Note: Our answer is longer than both our other sides... which is good because the hypotenuse is supposed to be the longest side!

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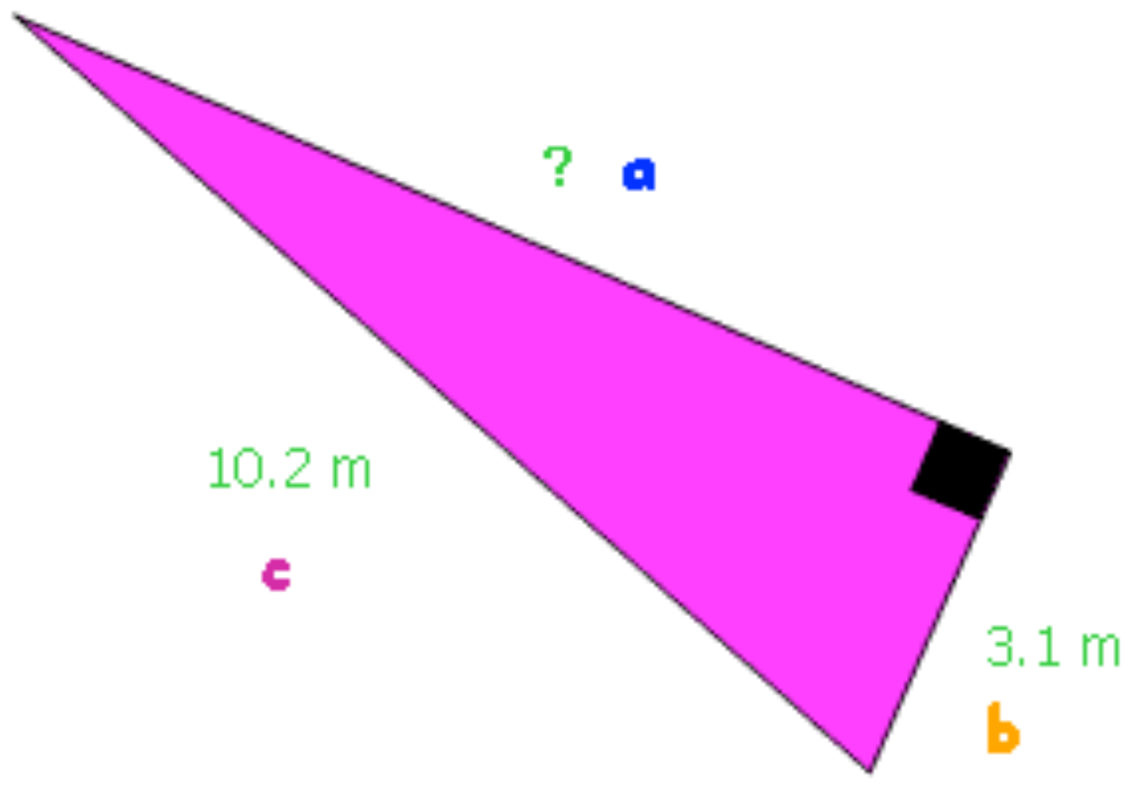
2.



Okay, so the side we want to find is NOT the Hypotenuse, so let's go through our routine:

1. Label the sides

2. Use the formula: $a^2 = c^2 - b^2$

3. Put in the numbers:

$$a^2 = 10.2^2 - 3.1^2$$

$$a^2 = 104.04^2 - 9.61^2$$

$$a^2 = 94.43$$

Square root both sides!

$$a = \sqrt{94.43}$$

$$a = 9.72m \quad \text{(2dp)}$$

Note: Our answer is shorter than our hypotenuse... which is good because the hypotenuse is supposed to be the longest side!

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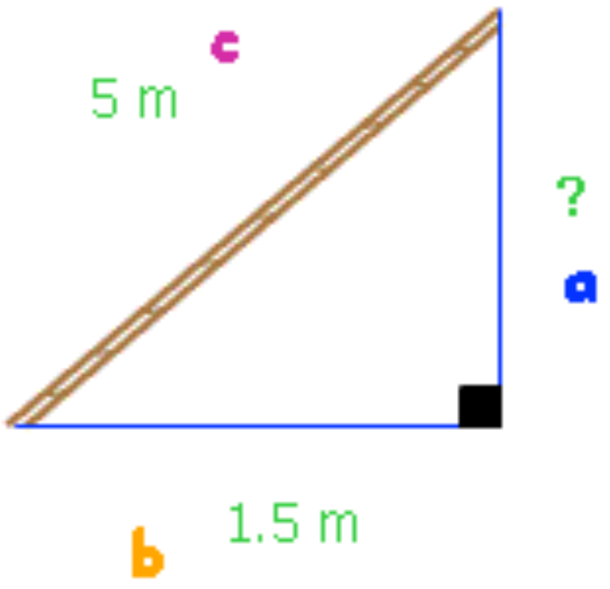
3.

A 5m ladder rests against the side of a house. The foot of the ladder is 1.5m away from the house. How far up the side of the house does the ladder reach?

At first glance this question does not appear to have anything to do with Pythagoras, but in these sort of situations, always follow this advice: IF IT'S TRICKY, DRAW A PICCY!!!... and then look what we have!



It's just a right angled triangle and we want to find a side that is NOT the Hypotenuse, so let's go through our routine:

1. Label the sides
2. Use the formula: $a^2 = c^2 - b^2$
3. Put in the numbers:

$$a^2 = 5^2 - 1.5^2$$

$$a^2 = 25^2 - 2.25^2$$

$$a^2 = 22.75$$

Square root both sides!

$$a = \sqrt{22.75}$$

$$a = 4.77m \quad \text{(2dp)}$$

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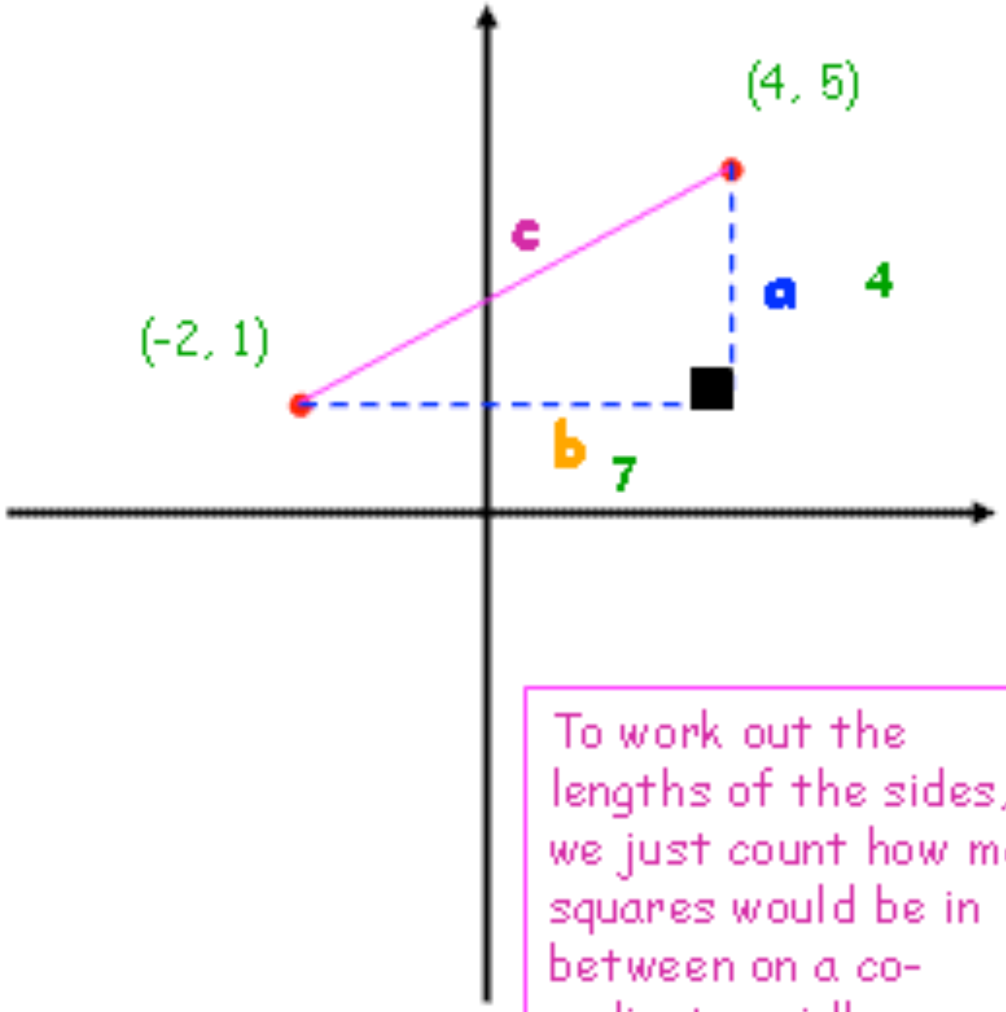
4. Find the distance between these two co-ordinates: (4, 5) and (-2, 1)

Again, at first glance this question does not appear to have anything to do with Pythagoras, but if we do a quick sketch of our co-ordinates, then look what we have!



To work out the lengths of the sides, we just count how many squares would be in between on a co-ordinate grid!

It's just a right angled triangle and we want to find the Hypotenuse, so let's go through our routine:

1. Label the sides
2. Use the formula: $c^2 = a^2 + b^2$
3. Put in the numbers:

$$c^2 = 4^2 + 7^2$$

$$c^2 = 16 + 49$$

$$c^2 = 65$$

$$c = \sqrt{65}$$

$$c = 8.1 \quad (1\text{dp})$$

Square root both sides!

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2. Sin, Cos and Tan

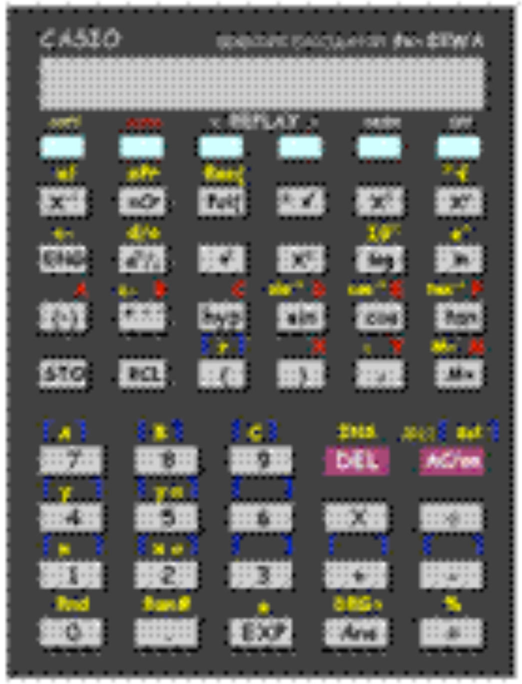
1. The Crucial Point about Sin, Cos and Tan

Just like Pythagoras Theorem, all the work we will be doing with Sin, Cos and Tan only works with RIGHT-ANGLED TRIANGLES

So... if you don't have a right-angled triangle, you might just have to add a line or two to make one!

2. Checking your Calculator is in the Correct Mode

Every now and again calculators have a tendency to do stupid things, one of which is slipping into the wrong mode for sin, cos and tan questions, giving you a load of dodgy answers even though you might be doing everything perfectly correctly!



Here is the check: Work out: sin 30

sin 3 0 =

And if you get an answer of **0.5**, you are good to go!

If not, you will need to change into degrees (DEG) mode.

Each calculator is different, but here's how to do this on mine:

MODE MODE 1

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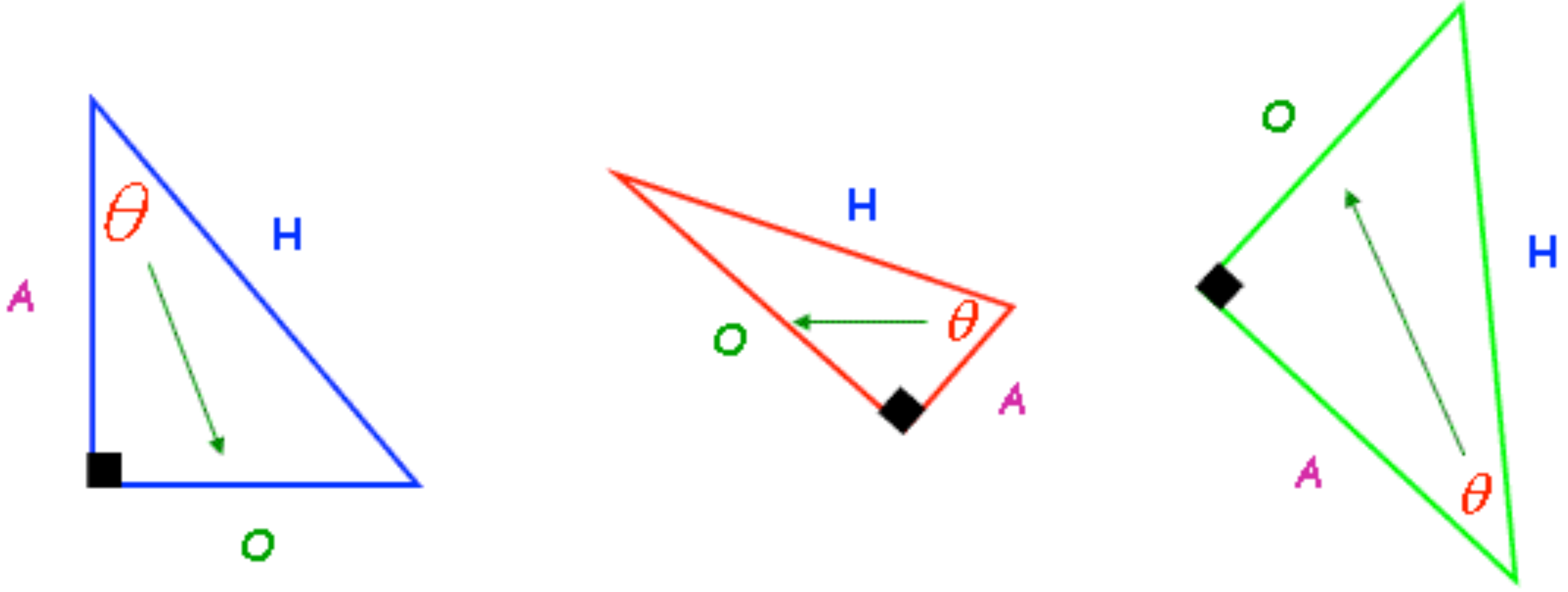
3. Labelling the Sides of a Right-Angled Triangle

- Before you start frantically pressing buttons on your calculator, you must work out which one of the trig ratios (sin, cos or tan) than you need, and to do this you must be able to label the sides of your right-angled triangle correctly.

- This is the order to do it:
  1. Hypotenuse (H) – the longest side, opposite the right-angle
  2. Opposite (O) – the side directly opposite the angle you have been given / asked to work out
  3. Adjacent (A) – the only side left!



Note: θ is just the Greek letter Theta, and it is used for unknown angles, just like x is often used for unknown lengths!

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4. The Two Ways of Solving Trigonometry Problems

Both methods start off the same:

1. Label your right-angled triangle
2. Tick which information (lengths of sides, sizes of angles) you have been given
3. Tick which information you have been asked to work out
4. Decide whether the question needs **sin**, **cos** or **tan**

The difference comes now, where you actually have to go on and get the answer.

Both of the following methods are perfectly fine, just choose the one that suits you best!

(a) Use the Formulas and Re-arrange

If you are comfortable and confident re-arranging formulas, then this method is for you!

Just learn the following formulas:

$$\text{Sine } \theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\text{Sin } \theta = \frac{O}{H}$$

$$\text{Cosine } \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{Cos } \theta = \frac{A}{H}$$

$$\text{Tangent } \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\text{Tan } \theta = \frac{O}{A}$$

Now just substitute in the two values you do know, and re-arrange the equation to find the value you don't know!

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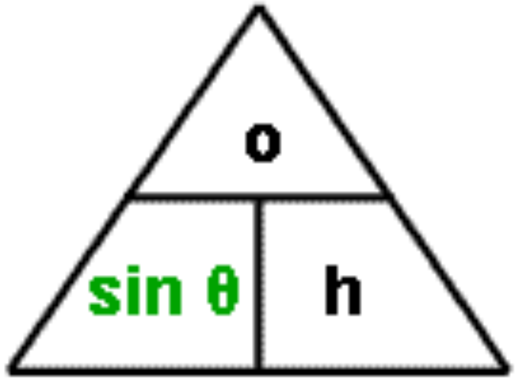
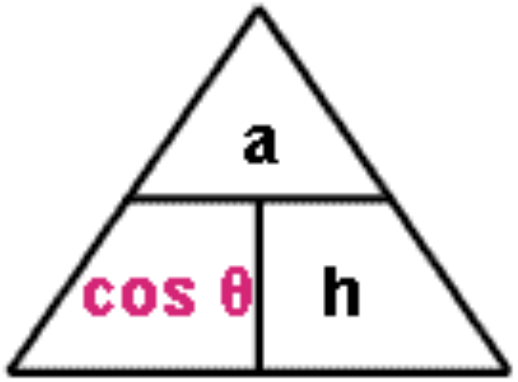
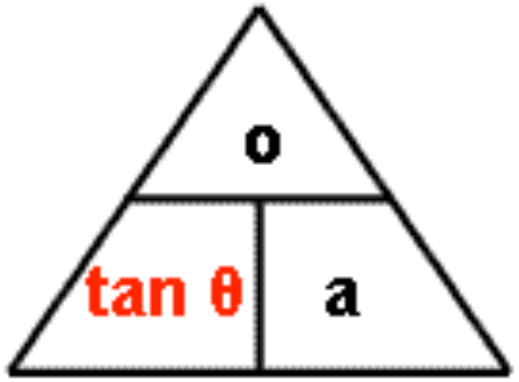
(b) Use the Formula Triangles

This is a clever little way of solving any trig problem.
Just make sure you can draw the following triangles from memory:





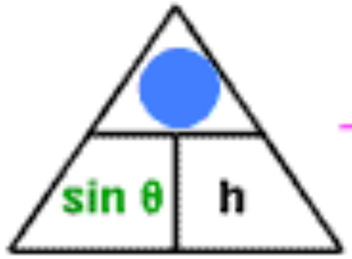


A good way to remember these is to use the initials, reading from left to right:

S O H **C A H** **T O A**

And make up a way of remembering them (a pneumonic, is the posh word!). Now, I know a good one about a horse, but it might be a bit too rude for this website...

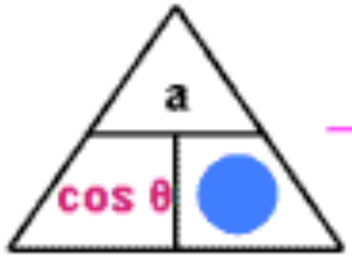
Anyway, once you have decided whether you need **sin**, **cos** or **tan**, just put your thumb over the thing (angle or side) you are trying to work out, and the triangle will magically tell you exactly what you need to do!



→

Finding Opposite:

$o = \sin\theta \times h$



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Finding Hypotenuse:

$h = a \div \cos\theta$

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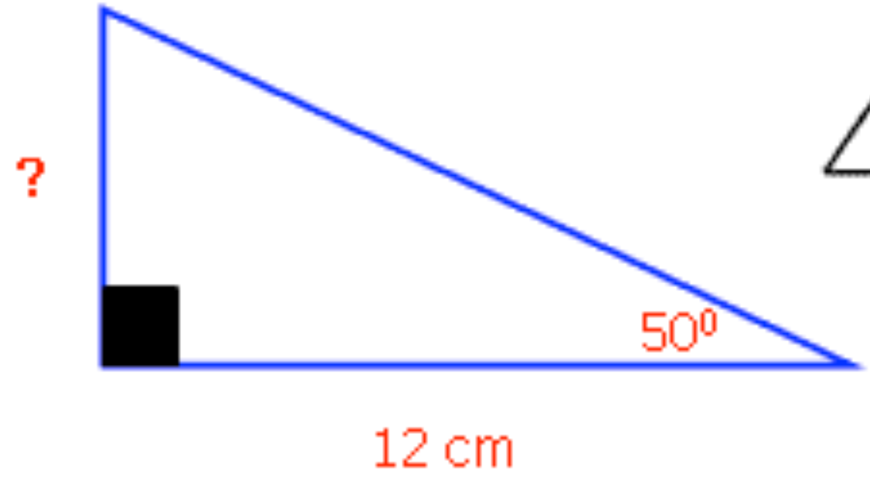
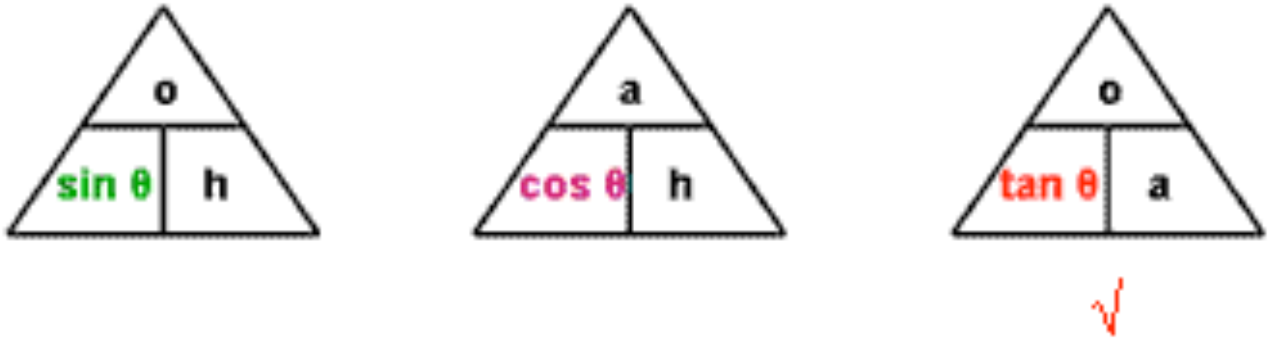
Examples

1.





Okay, here we go:

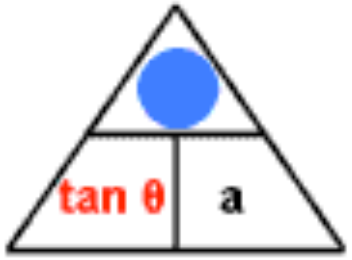
1. Label the sides

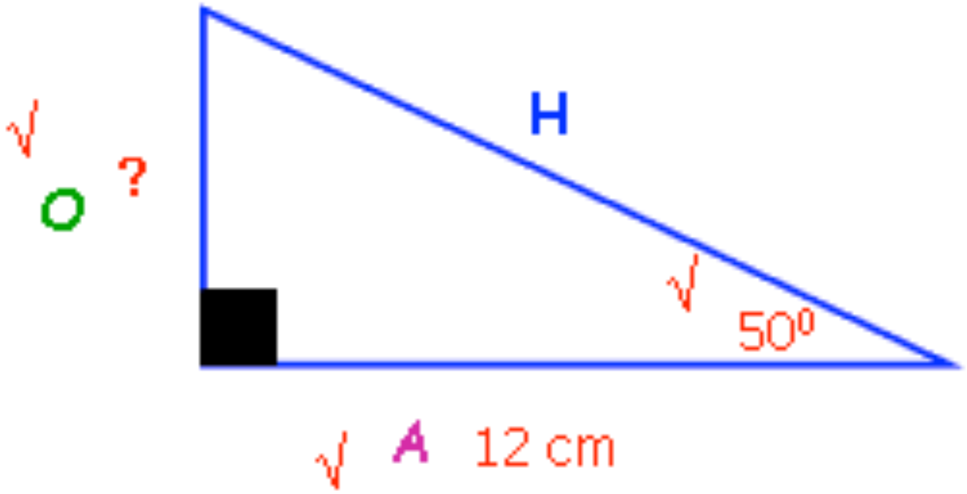
2. Tick which information we have been given… which I reckon is the angle and the Adjacent

3. Tick which information we need… which I reckon is the Opposite side

4. Decide whether we need **sin**, **cos** or **tan** … well, looking above, the only one that contains both O and A is… **Tan**!

5. Now we place our thumb over the thing we need to work out, which is the Opposite:





6. And now we know how to do it!

o = Tan θ **x** a

14.3 cm (1dp)

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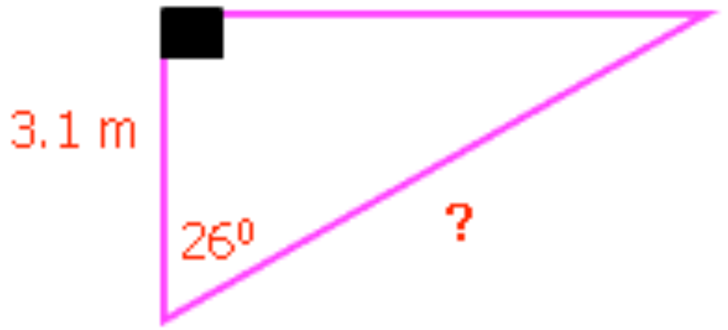
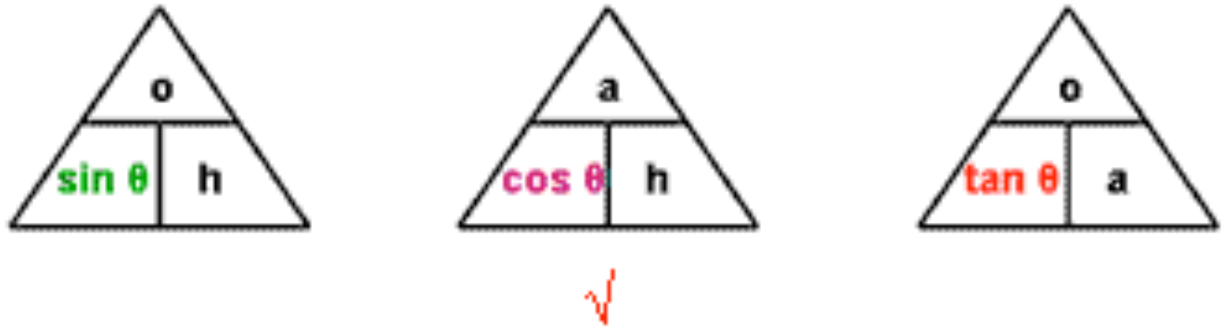
2.





Okay, here we go:

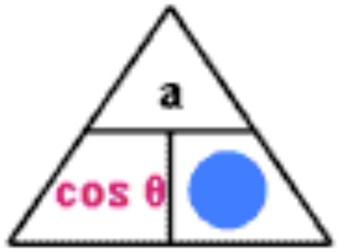
1. Label the sides

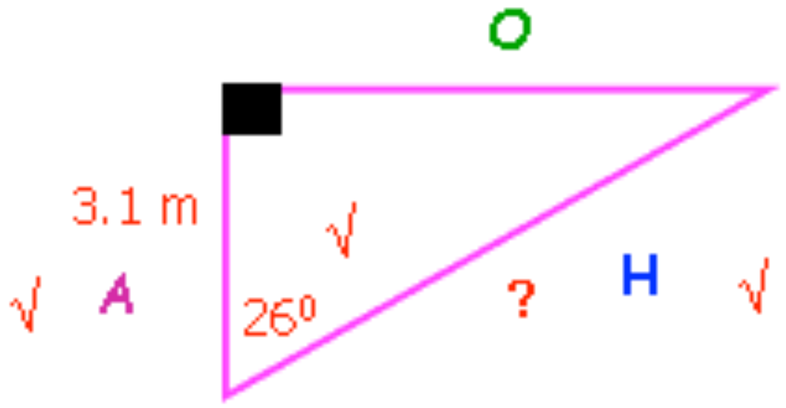
2. Tick which information we have been given… which I reckon is the angle and the Adjacent

3. Tick which information we need… which I reckon is the Hypotenuse side

4. Decide whether we need **sin**, **cos** or **tan** … well, looking above, the only one that contains both A and H is… **Cos**!

5. Now we place our thumb over the thing we need to work out, which is the Hypotenuse:





6. And now we know how to do it!

$$h = a \div \text{Cos}\,\theta$$

3 . 1 ÷ (cos 2 6) =

3.45 m (2dp)

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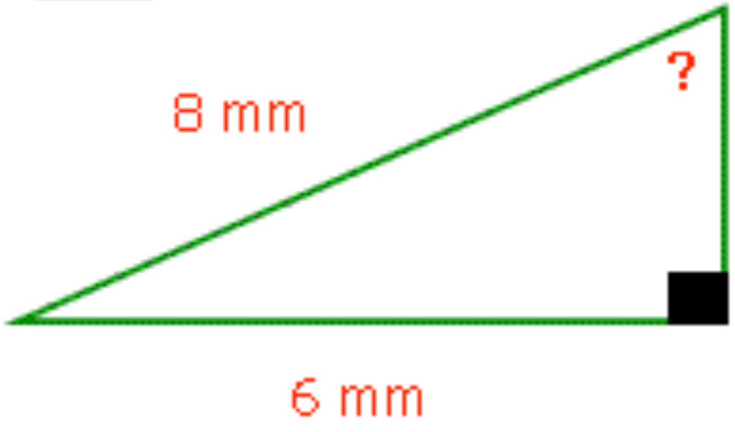
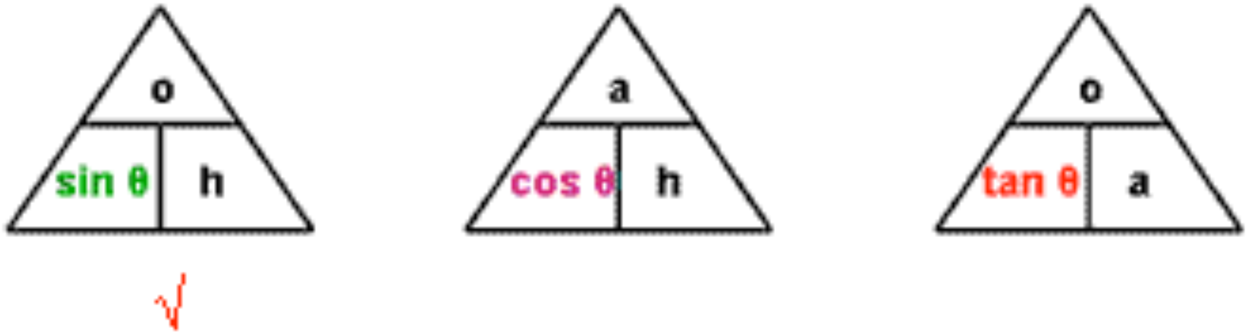
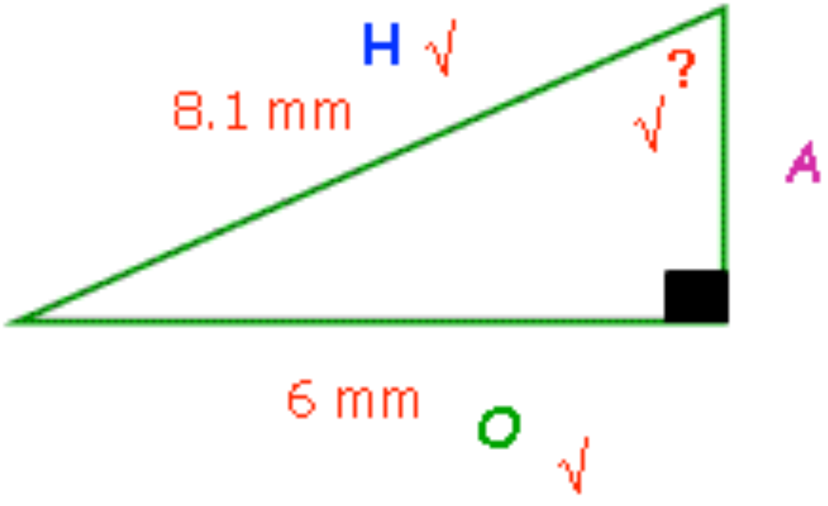
3.







Okay, here we go:

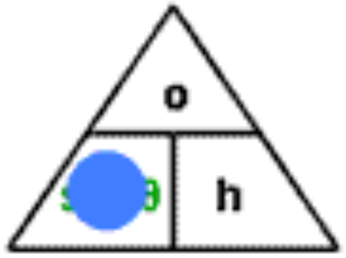
1. Label the sides

2. Tick which information we have been given... which I reckon is the Hypotenuse and the Opposite

3. Tick which information we need... which I reckon is the angle

4. Decide whether we need **sin**, **cos** or **tan** ... well, looking above, the only one that contains both O and H is... **Sin**!

5. Now we place our thumb over the thing we need to work out, which is the angle... or Sin θ



6. Okay, be careful here:

Sin θ = o ÷ h

Sin θ = 0.740740740740...

But that's not the answer! We don't want to know what Sin θ is, we want to know what θ is, so we must use "inverse sin" to leave us with just θ on the left hand side:

47.79^{0} (2dp)

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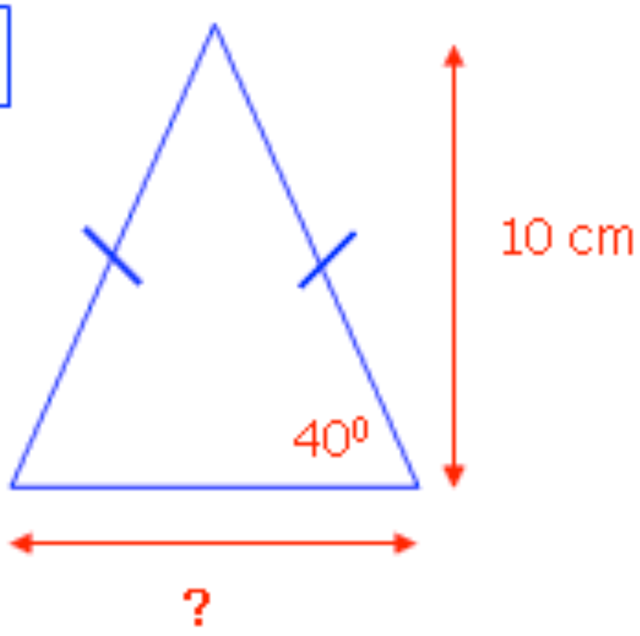
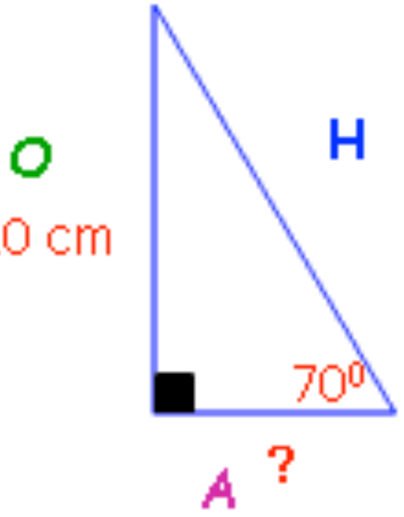
4.



Now, we have a problem here… we don't have a right-angled triangle! But we can easily make one appear from this isosceles triangle by adding a vertical line down the centre, and then we can carry on as normal…



1. Label the sides

2. Tick which information we have been given… which I reckon is the angle and the Opposite

3. Tick which information we need… which I reckon is the Adjacent

4. Decide whether we need **sin**, **cos** or **tan** … well, looking back, the only one that contains both O and A is… **Tan**!

5. Now we place our thumb over the thing we need to work out, which is the Adjacent

6. And now we know how to do it!

$$a = o \div \text{Tan}\,\theta$$

1 0 ÷ (tan 7 0) =

3.639702…

But that's not the answer! We've only worked out half of the base of the isosceles triangle! So we need to double this to give us our true answer of:

7.28 cm (2dp)

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3. 3D Trigonometry

The Secret to Solving 3D Trigonometry Problems

- 3D Trigonometry is just the same as bog-standard, flat, normal trigonometry

- All we need are the skills we learnt in the last two sections:
  1. Pythagoras
  2. Sin, Cos and Tan

- The only difference is that is a little bit harder to spot the right-angled triangles

- But once you spot them:
  - Draw them out flat
  - Label your sides
  - Fill in the information that you do know
  - Work out what you don't in the usual way!

- And if you can do that, then you will be able to tick another pretty tricky topic off your list!

Please Remember: You need a right-angled triangle to be able to use either Pythagoras or Sin, Cos and Tan... and I promise that will be the last time I say it!

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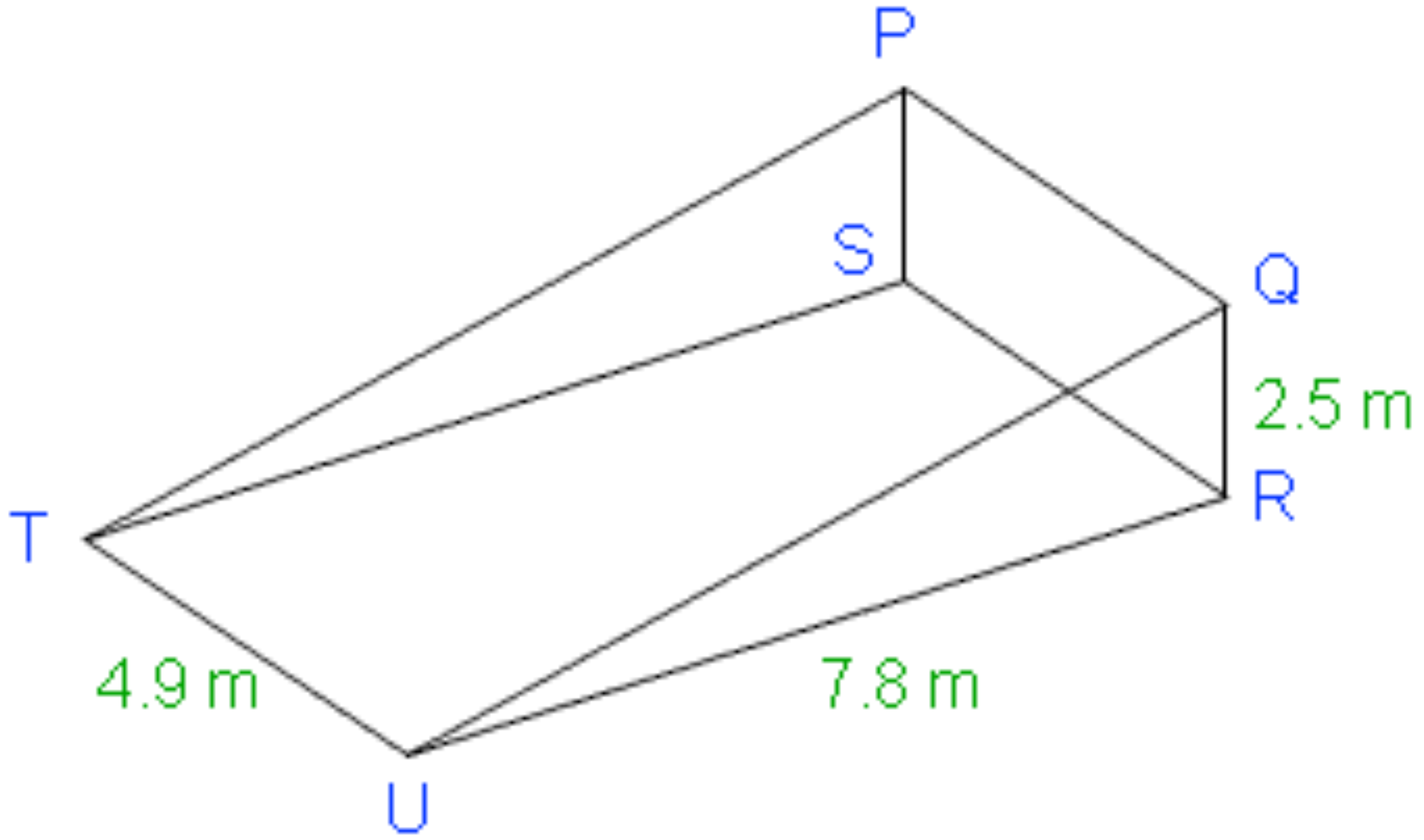
Example 1

The diagram below shows a record breaking wedge of Cheddar Cheese in which rectangle PQRS is perpendicular (at 90^0 to) to rectangle RSTU. The distances are shown on the diagram.

Calculate: (a) The distance QT (b) The angle QTR



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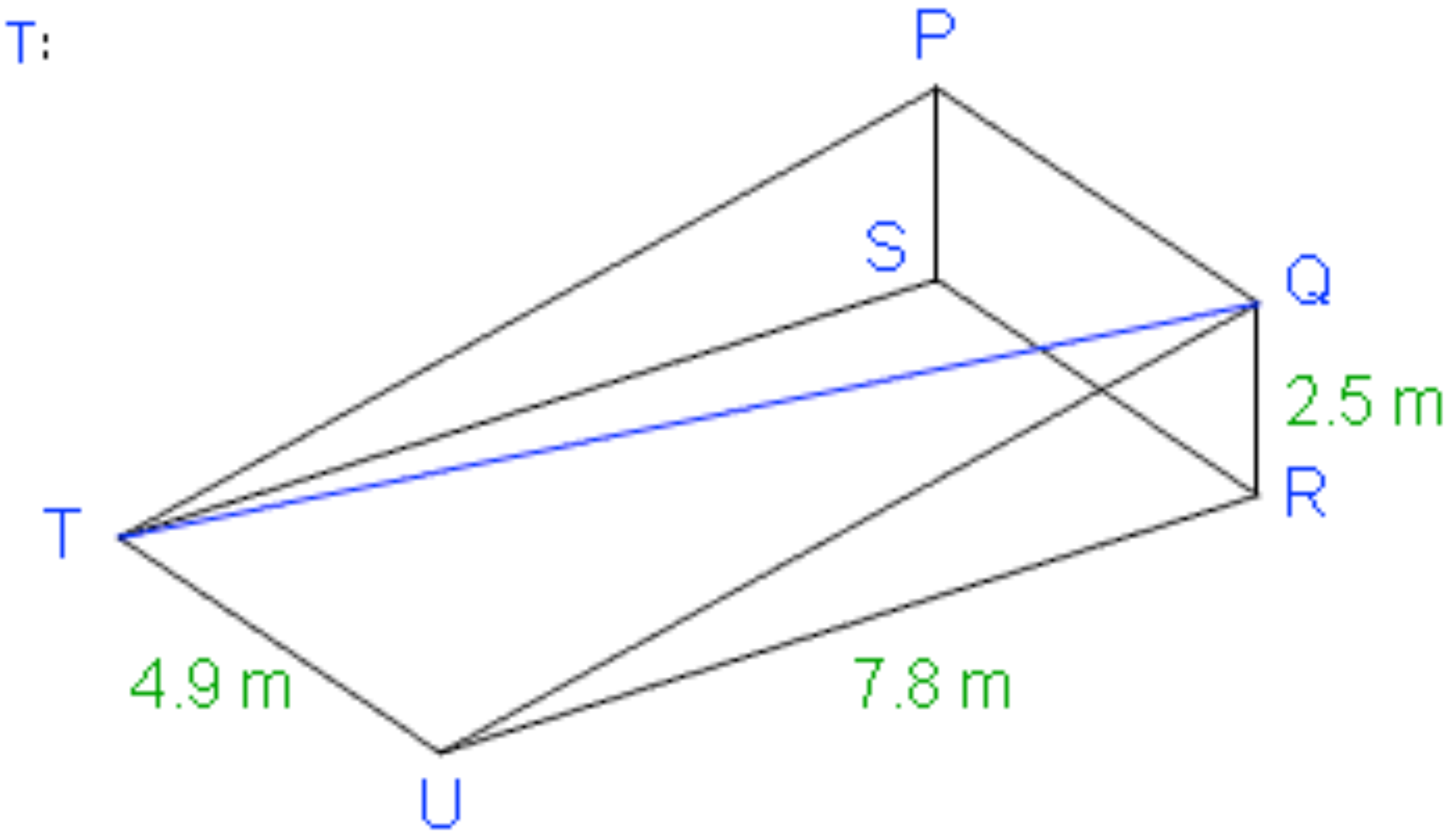
Working out the answer (a):

The first thing we need to figure out is what we are actually trying to work out!

We need the line QT:



Now, as I said, the key to this is spotting the right-angled triangles...

Well, I can see a nice one: TQR.

That contains the length we want, and we already know how long QR is...

So now all we need to do is work out length TR...

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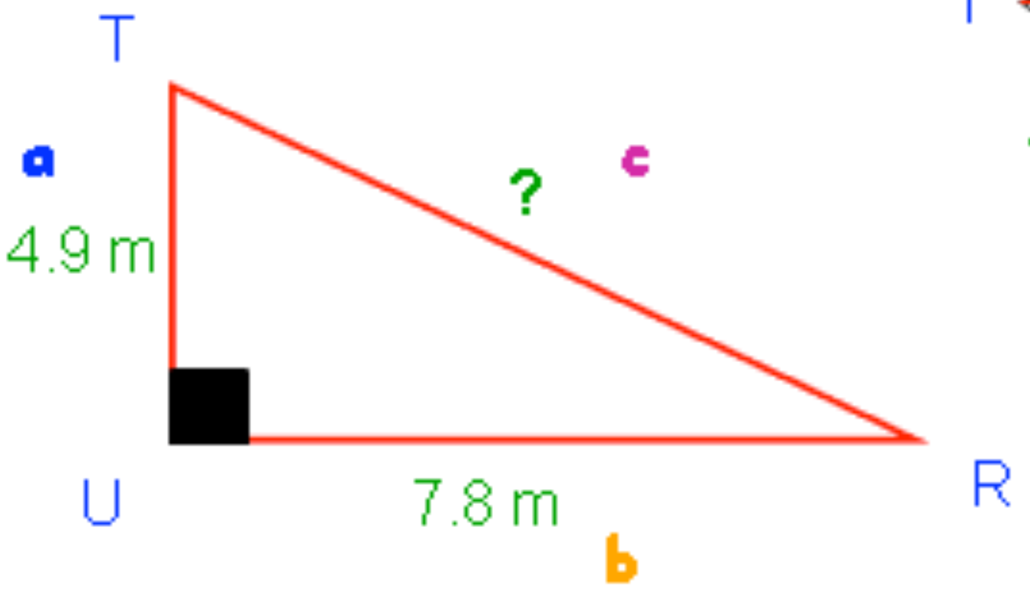
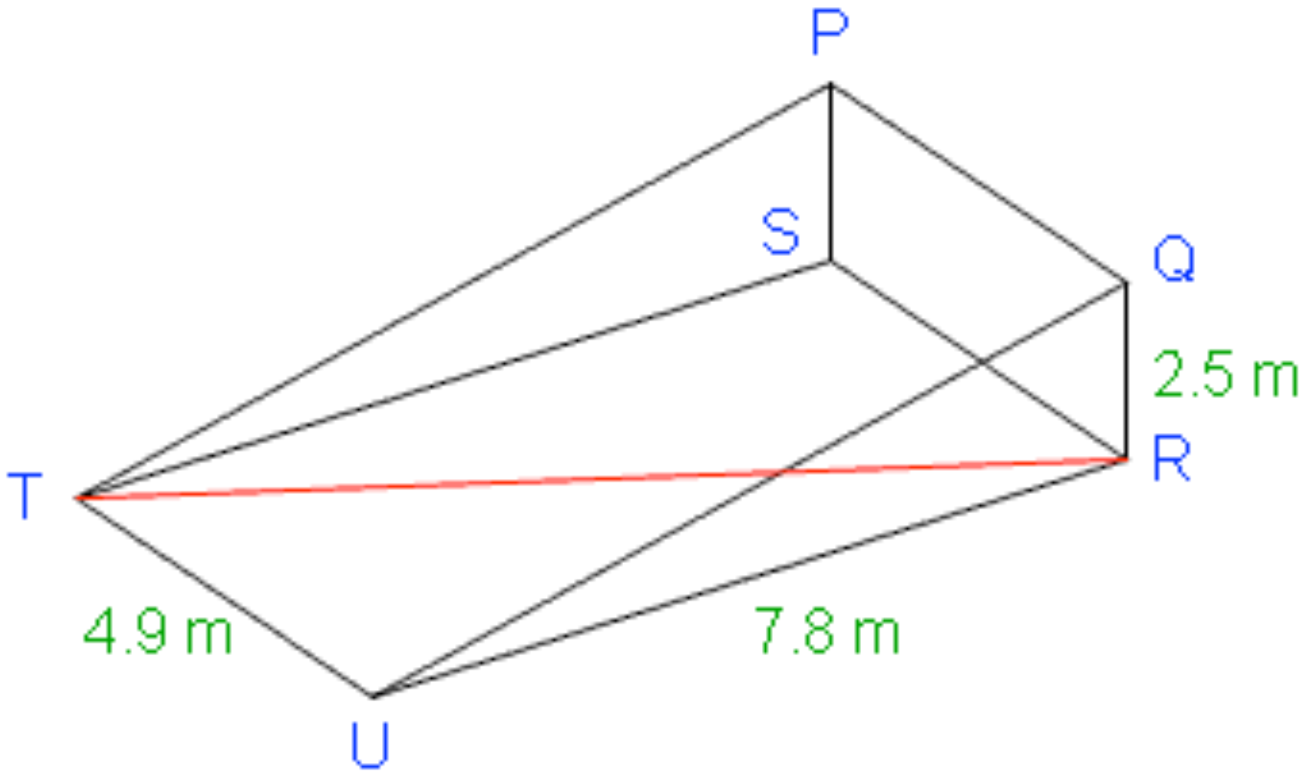
Working Out TR:

Okay, if you look carefully, you should be able to see a right-angled triangle on the base of this wedge of cheese

It's the triangle TRU:





Well, we have two sides and we want to work out the Hypotenuse… This looks like a job for Pythagoras!

1. Label the sides

2. Use the formula: $c^2 = a^2 + b^2$

3. Put in the numbers:

$$c^2 = 4.9^2 + 7.8^2$$

$$c^2 = 24.01 + 60.84$$

$$c^2 = 84.85$$

$$c = \sqrt{84.85}$$

$$c = 9.211\ldots m$$

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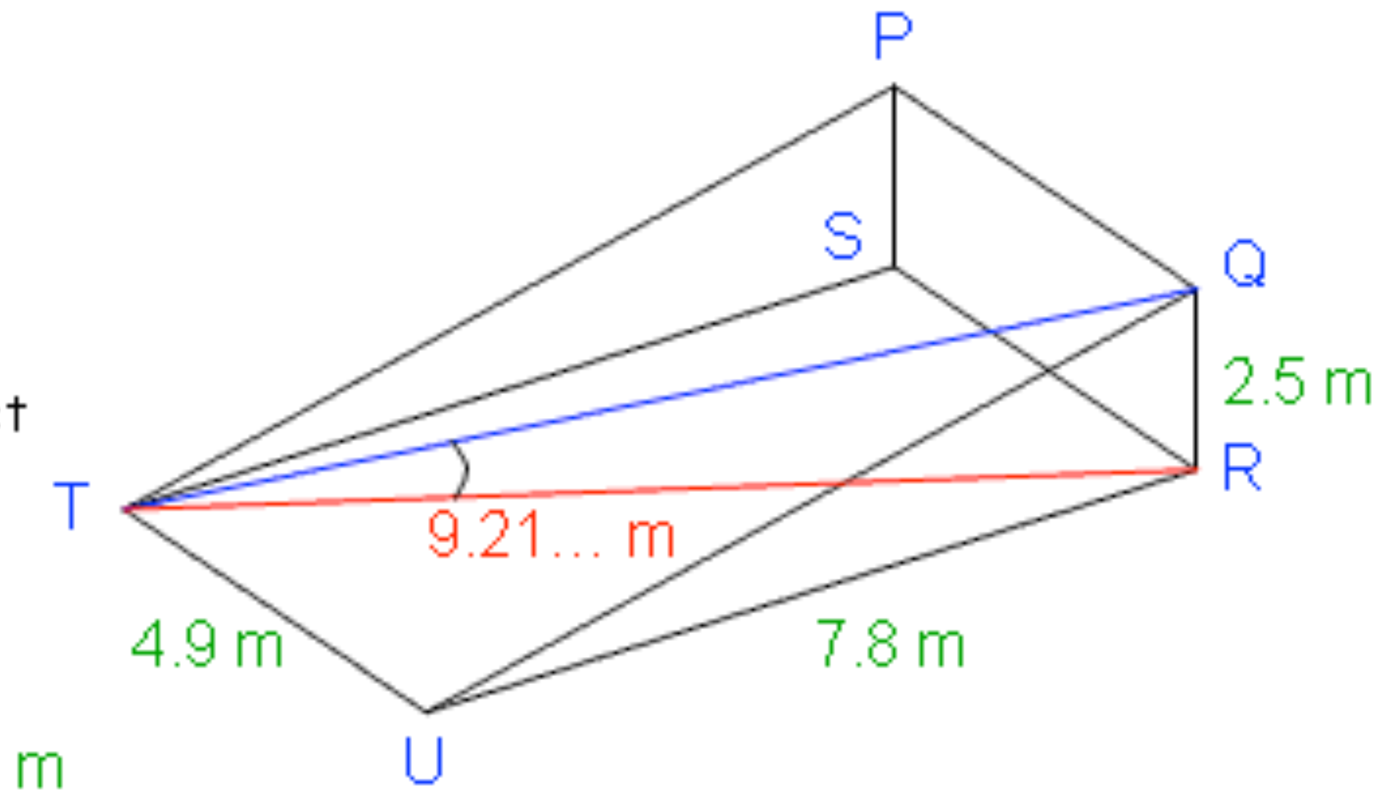
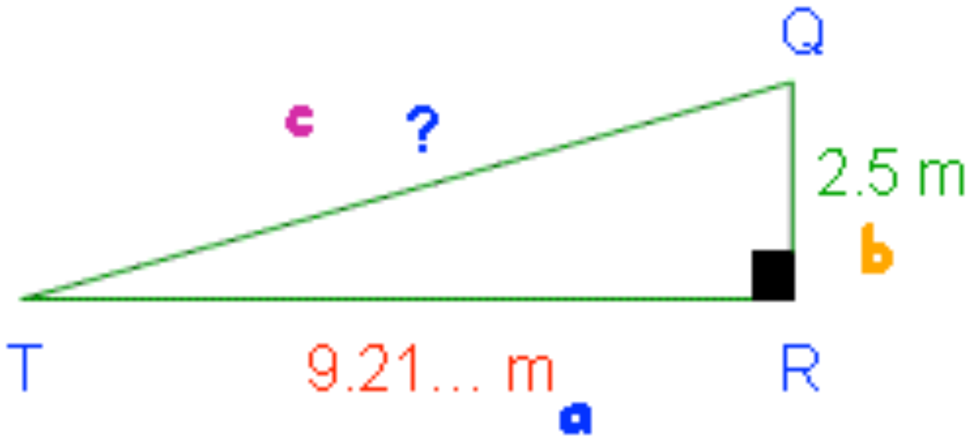
Working Out TQ:

Okay, so now we have all we need to be able to calculate TQ.

Just make sure you draw the correct right-angled triangle!





Once again, we have two sides and we want to work out the Hypotenuse... This looks like a job for Pythagoras!

1. Label the sides

2. Use the formula: $c^2 = a^2 + b^2$

3. Put in the numbers:

$$c^2 = 9.21...^2 + 2.5^2$$

$$c^2 = 84.85 + 6.25$$

$$c^2 = 91.1$$

$$c = \sqrt{91.1}$$

$$c = 9.54m \quad \text{(2dp)}$$

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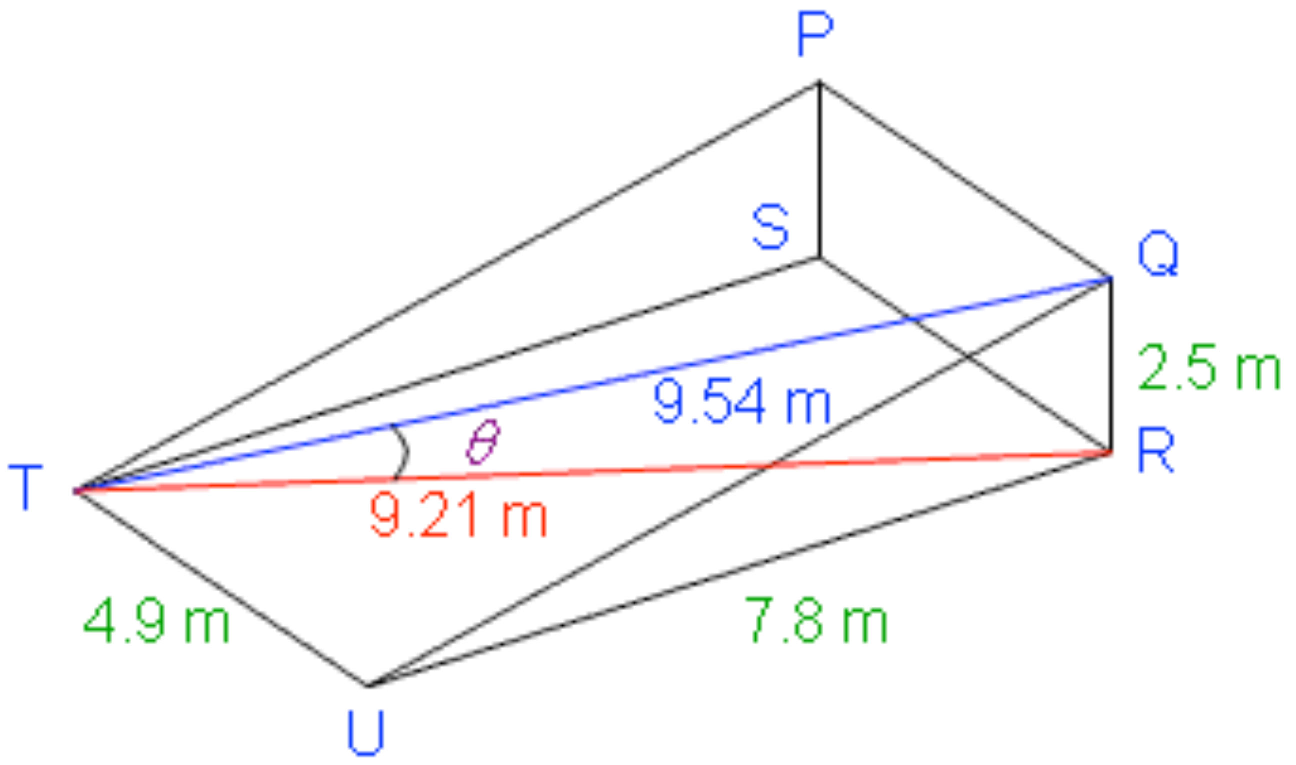
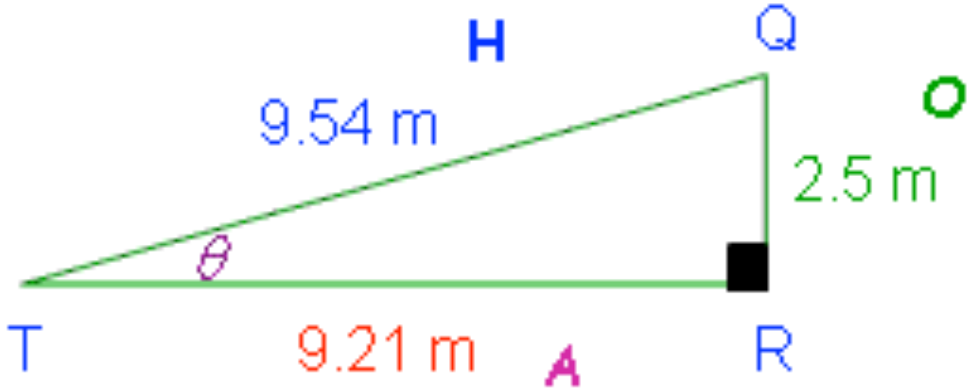
Working out the answer (b):

Again, we must be sure we know what angle the question wants us to find!

I have marked angle QTR on the diagram



So now we draw our right-angled triangle:



To calculate the size on an angle, we must use either **sin**, **cos** or **tan**, which means first we must label our sides!

Now, because we actually know all three lengths, we can choose! I'm going for **tan**!

Tan θ = o ÷ a

2 . 5 ÷ 9 . 2 1 =

Tan θ = 0.27144...

SHIFT tan⁻¹ F tan Ans =

15.2^{0} (1dp)

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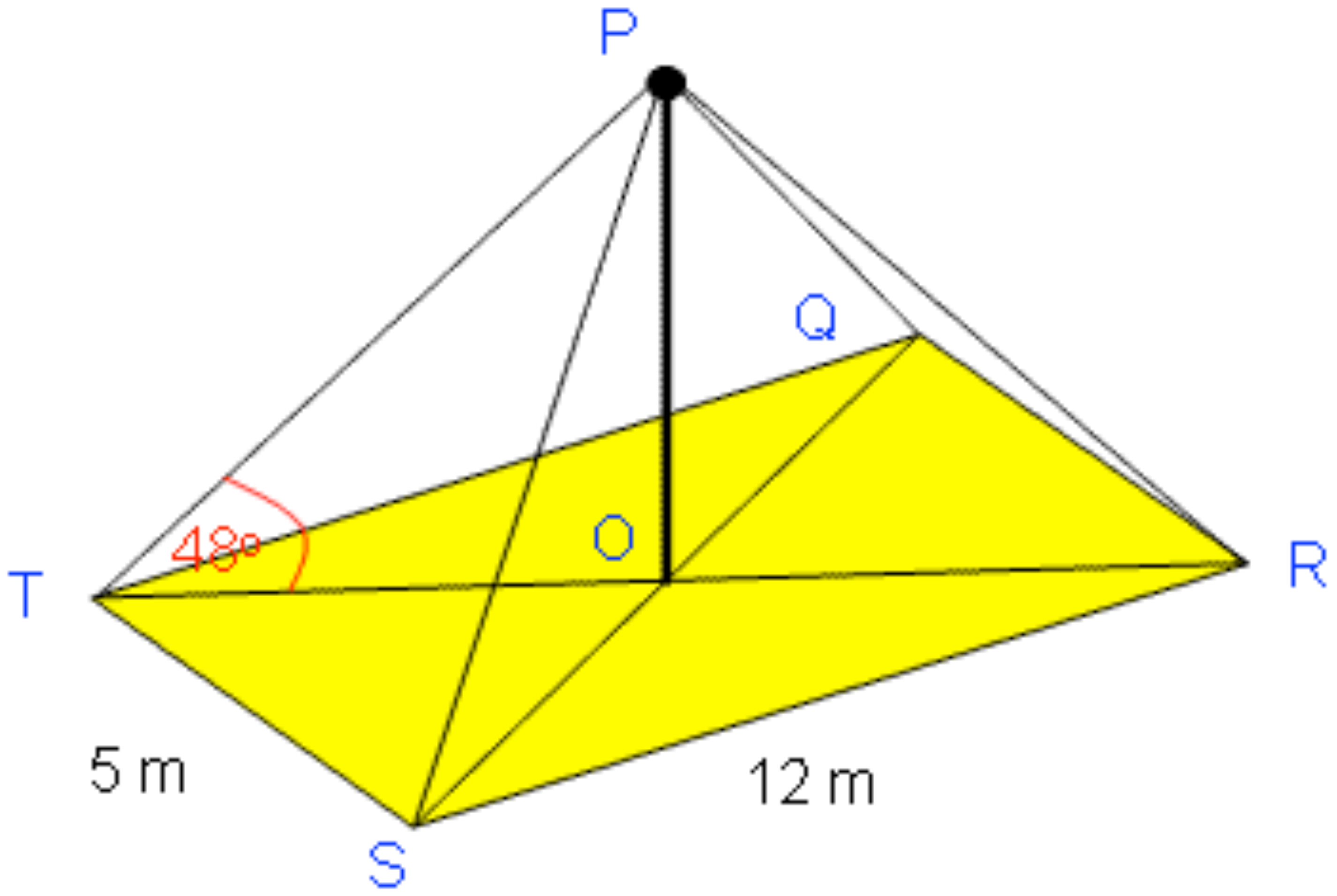
Example 2

The diagram below shows a plan of a tent that I am trying to erect before the rain comes. OP is a vertical pole, and O is at the very centre of the rectangle QRST. The lengths and angles are as shown on the diagram. Calculate the height of the vertical pole OP.



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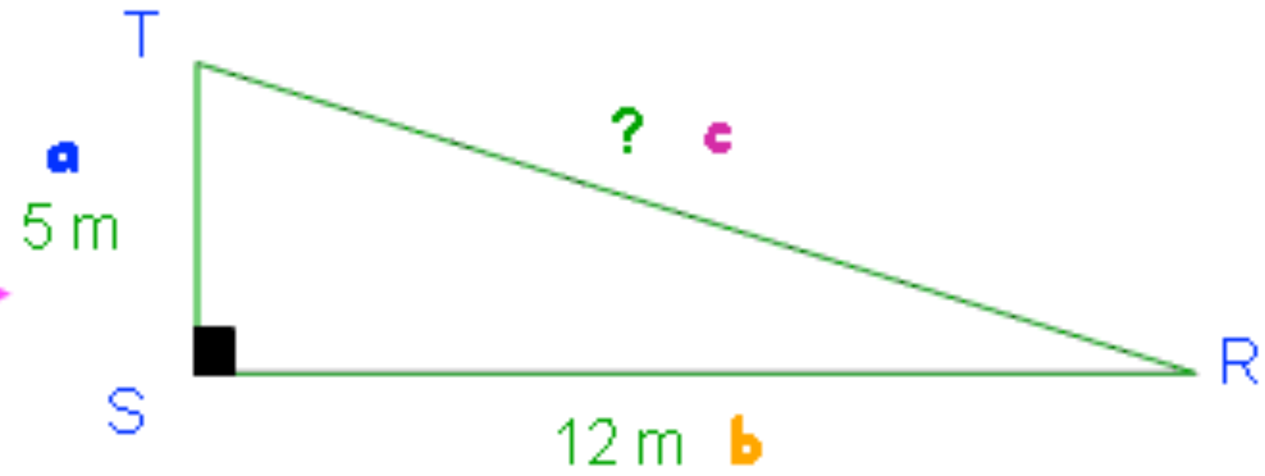
1. Working Out OT:

You should be able to see that if we can work out OT, we will then have a right-angled triangle which will give us OP!

So, to get started we need to use the base of the rectangle:

Well, OT is half way along the line TR line, so it must be... **6.5m**



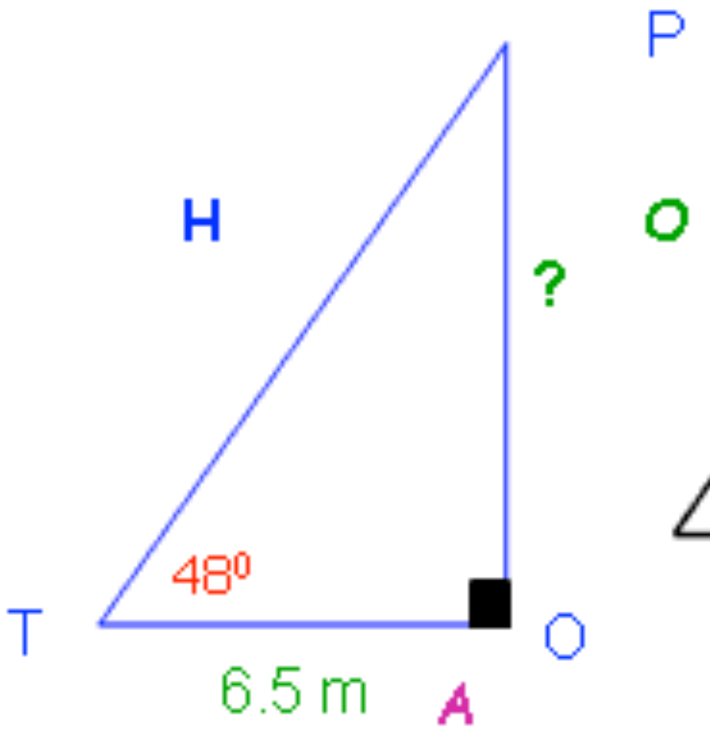
$$c^2 = 5^2 + 12^2$$

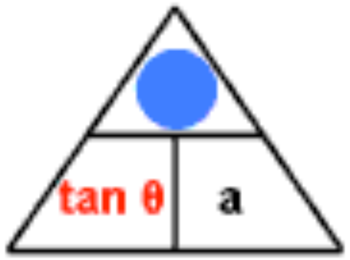
$$c^2 = 25 + 144$$

$$c^2 = 169$$

$$c = \sqrt{169}$$

$$c = 13m$$





o = Tan θ **x** a

(tan 4 8) X 6 . 5 =

7.22 m (2dp)

2. Working Out OP:

And now we have a right-angled triangle where we know one length (TR), and we know one angle (OTP)... so we can work out any side using a bit of **sin**, **cos** or **tan**!

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4. Sine and Cosine Rules

The Big Problem with Trigonometry

- As far as mathematical things go, Pythagoras, and the trio of Sin, Cos and Tan, were pretty good... weren't they?

- However, they had one major draw back...

They only worked for right-angled triangles!

- That certainly limited their use.

- Well, imagine if we had some rules which worked for... wait for it... any triangle!

- Well, you'll never guess what... we do!... The Sine and Cosine Rules!

The Crucial Point about the Sine and Cosine Rules

You must know when to use each rule... what information do you need to be given?

If you can get your head around that, then it's just plugging numbers into formulas!

Note: In all the formulas, small letters represent sides, and Capital Letters represent Angles!

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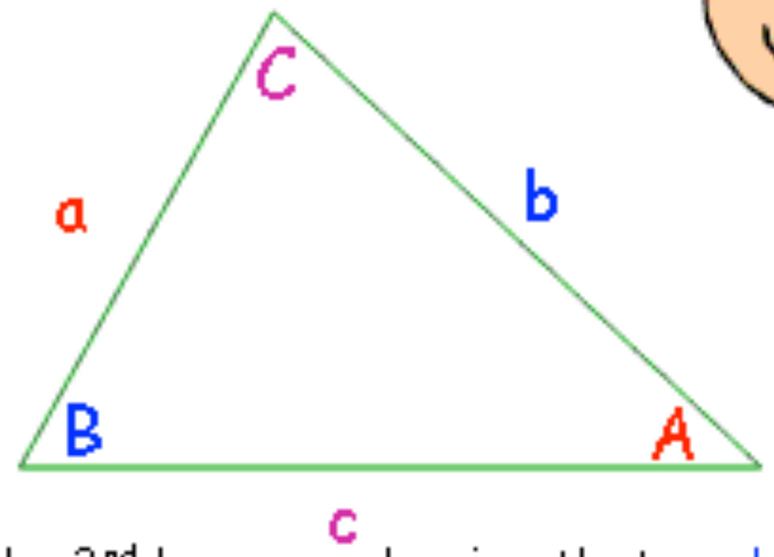
1. The Sine Rule – Finding an unknown Side

What Information do you need to be given?

Two angles and the length of a side

What is the Formula?

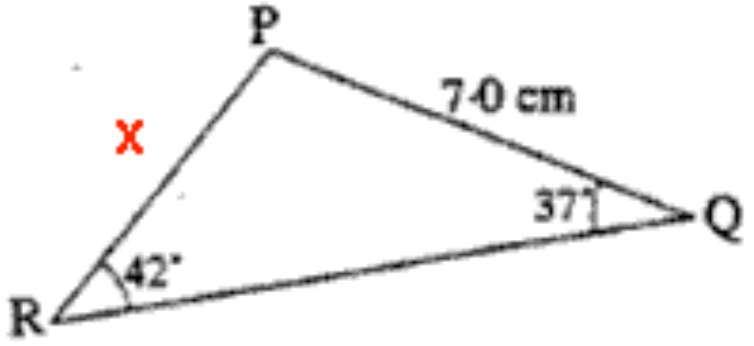
$$\boxed{\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC}}$$



Remember:

If you are given two angles, you can easily work out the 3rd by remembering that angles in a triangle add up to 180⁰!

Example



$$\frac{a}{SinA} = \frac{b}{SinB}$$

$$\frac{x}{Sin37} = \frac{7.0}{Sin42}$$

$$x = \frac{7.0}{Sin42} \times Sin37$$

Multiply both sides by sin 37

$$x = 6.3cm \ (1dp)$$

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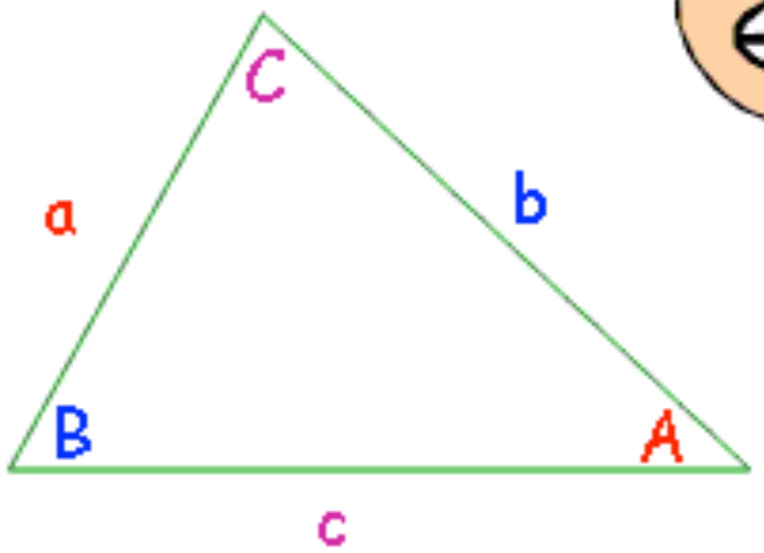
2. The Sine Rule – Finding an unknown Angle

What Information do you need to be given?

Two lengths of sides and the angle NOT INCLUDED
(i.e. not between those two sides!)

What is the Formula?

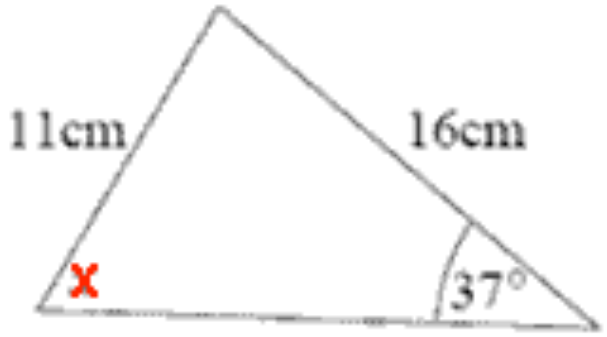
$$\boxed{\frac{SinA}{a} = \frac{SinB}{b} = \frac{SinC}{c}}$$



Remember:

If the angle is included, you will have to use the Cosine Rule!

Example



$$\frac{SinA}{a} = \frac{SinB}{b}$$

$$\frac{Sinx}{16} = \frac{Sin37}{11}$$

$$Sinx = \frac{Sin37}{11} \times 16$$

Multiply both sides by 16

$$Sinx = 0.8753... \rightarrow x = 61.1^0 \ (1dp)$$

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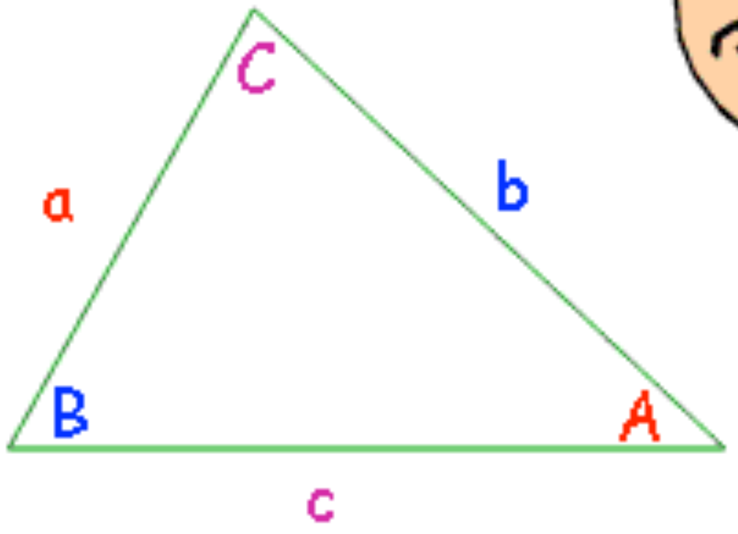
3. The Cosine Rule – Finding an unknown Side

What Information do you need to be given?

Two sides of the triangle and the INCLUDED ANGLE
(i.e. the angle between the two sides!)

What is the Formula?

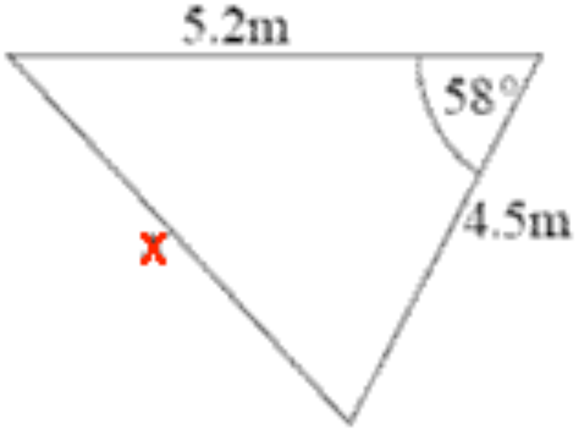
$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$



Remember:

You must be pretty good on your calculator to get these ones correct!

Example



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$x^2 = 5.2^2 + 4.5^2 - 2 \times 5.2 \times 4.5 \times \cos 58$$

$$x^2 = 5.2^2 + 4.5^2 - 2 \times 5.2 \times 4.5 \times \cos 58$$

$$x^2 = 22.48977...$$

$$x = \underline{4.74m}\ (2dp)$$

Square root both sides

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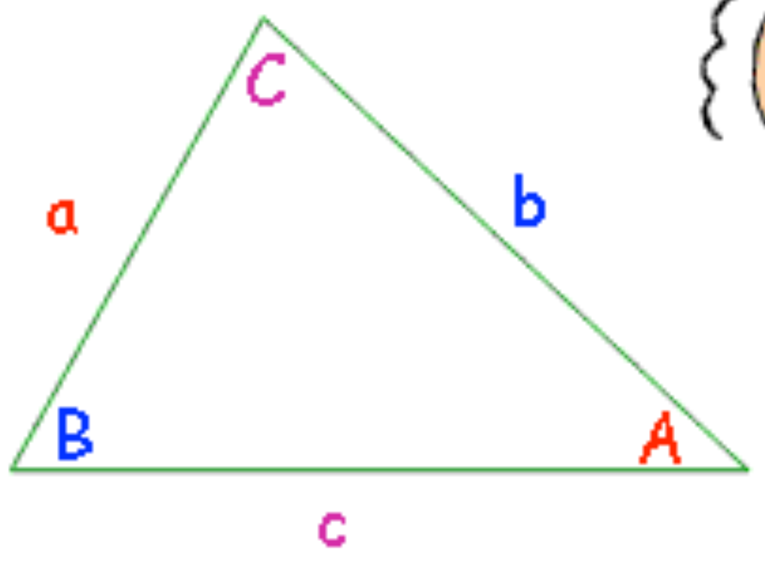
4. The Cosine Rule – Finding an unknown Angle

What Information do you need to be given?

All three lengths of the triangle must be given!



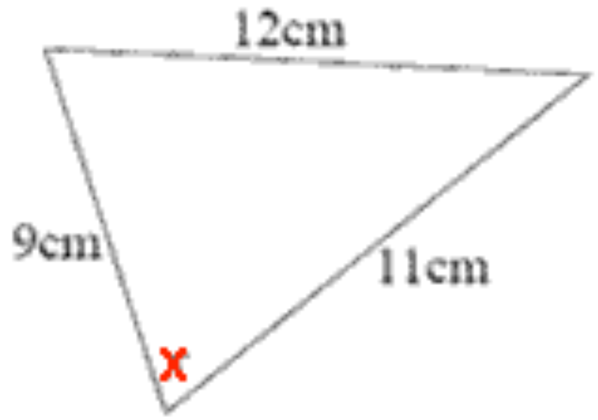
What is the Formula?

$$\boxed{CosA = \frac{b^2 + c^2 - a^2}{2bc}}$$

Remember:

This is just a re-arrangement of the previous formula, so you only need to remember one!

Example



$$CosA = \frac{b^2 + c^2 - a^2}{2bc}$$

$$Cosx = \frac{9^2 + 11^2 - 12^2}{2 \times 9 \times 11}$$

$$Cosx = \frac{58}{198}$$

$$Cosx = 0.292929\ldots \rightarrow x = 72.97^0 (2dp)$$

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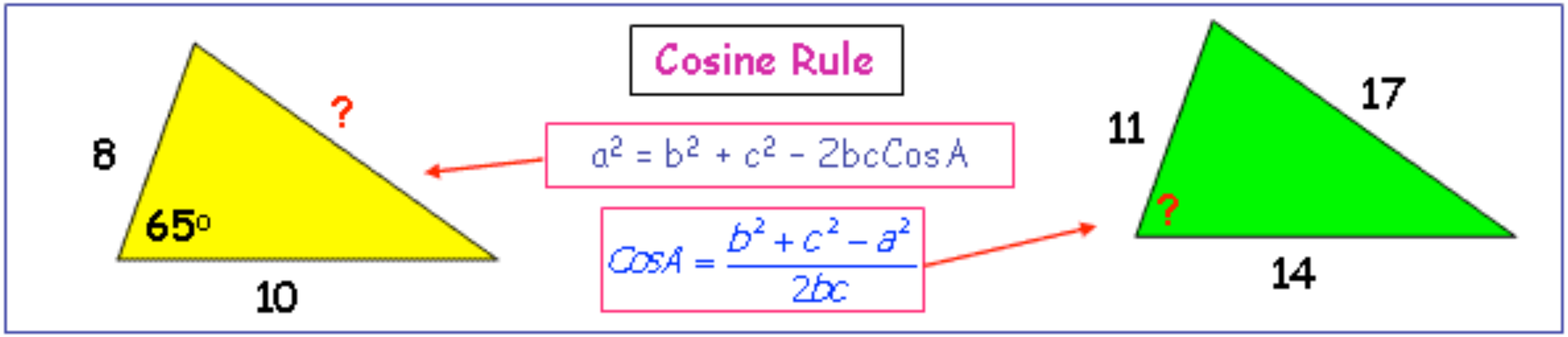
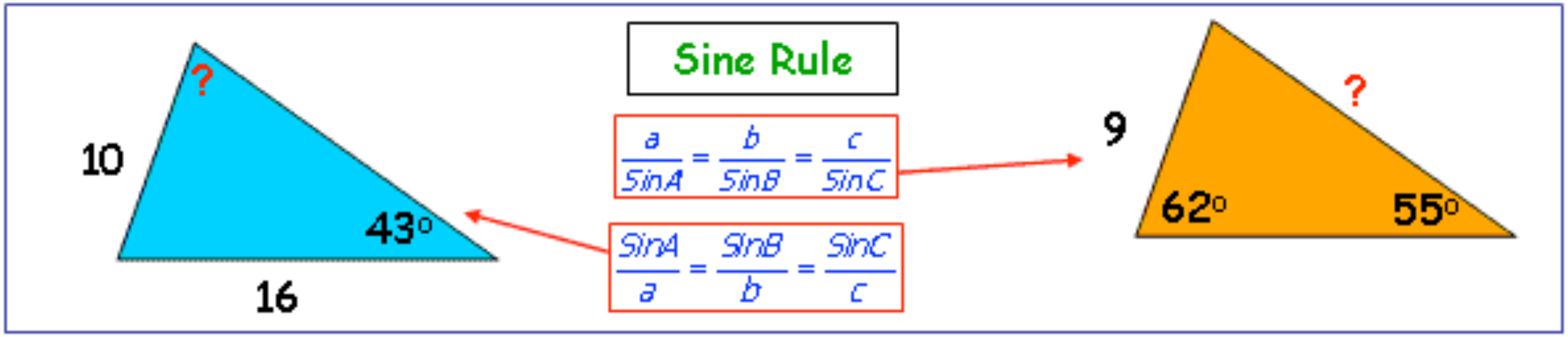
A Nice Little Summary



Cosine Rule

$a^2 = b^2 + c^2 - 2bc\cos A$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



Sine Rule

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

	Finding Sides	Finding Angles
Cosine Rule	Need 2 sides and included angle	Need all 3 sides
Sine Rule	Need 2 angles and any side	Need 2 sides and an angle not included

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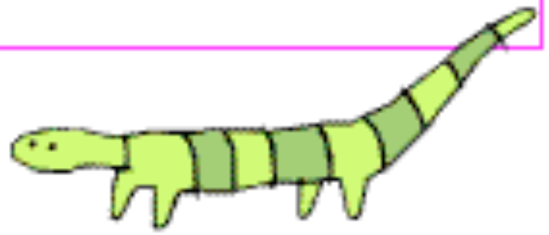
Data Handling and Probability

1. Probability

Mr Barton's Favourite!

I'll come clean straight away… probability is my favourite maths topic. Sad, hey? I'm not too good at drawing shapes or using a compass, but I'm pretty good at Probability, and I like it, and hopefully after reading this section, you will too!



What is Probability?…

- Probability is the likelihood or chance of something happening.
- And that something can be pretty much anything – from something boring like getting a head when you toss a coin, to something much more interesting, like the probability of a £300million space rocket returning safely from it's mission.
- We use probabilities every single day in the decisions we make without even knowing it.
- The tools I am going to arm you with in this section will hopefully enable you to understand and enjoy probability through to A Level and beyond…. that's the plan, anyway.

The Lingo you need:

Experiment – now, this doesn't necessarily mean rats and men in lab coats, it just means something is happening, and someone else is observing what happens

Outcomes – these are all the different things that could happen in a probability experiment. One question you must always ask yourself is: "is every outcome equally likely to happen?"

Event – this just means the particular outcome or outcomes we are interested in

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The most important maths formula you will ever learn...

Here it comes...

$$P(\text{event}) = \frac{\text{the number of ways the event could happen}}{\text{the total number of possible equally likely outcomes}}$$

What each bit means:

P (event) – this is just a quick way of writing: "the probability of an event happening".
e.g. P (rain on Wednesday) means "the probability it will rain on Wednesday"

The number of ways the event could happen – you have to carefully count up all the different ways there are of the event you are interested in actually occurring

The total number of possible equally likely outcomes – this is the hardest and most important bit. You must carefully count up all the total possible things that could happen, but you must remember that they must all be equally likely!

And when you get your answer using the formula, it will be a fraction, and you should simplify it you can!

And everything you need to know about probability comes from this formula!

Now, we will discover all the important probability concepts, using a few examples...

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Big Example 1

Imagine, for some reason, someone has put each of the 26 letters of the alphabet on identical tiles and chucked them in a bag. This person then decides it would be fun to get you to close your eyes and pick tiles out of this bag. You are not so sure, but you decide to give it a go as it will be a good way of learning about probability.

Question 1: What is the probability of picking out a vowel?

Well, let's use our formula:

Quick way of writing: "the probability of picking a vowel"

$$P\,(\mathit{vowel}) = \frac{5}{26}$$

The number of vowels in the bag: a, e, i, o, u

The number of equally likely things that could happen: we could have picked any of the 26 identical tiles!

Question 2: What is the probability of picking out a letter?

The answer might be obvious, but let's see why:

Quick way of writing: "the probability of picking a letter"

$$P\,(\mathit{letter}) = \frac{26}{26} = 1$$

There are 26 letters in the bag, and any will do for us!

There are 26 equally likely outcomes

Rule 1: If something has a probability of 1, it is CERTAIN to happen

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Question 3: What is the probability of picking out a number?

Again, it's easy, but look why it works!

There is nothing in the bag we are interested in as there are no numbers!

Quick way of writing: "the probability of picking a vowel"

$$P\,(number) = \frac{0}{26} = 0$$

The number of equally likely things that could happen: we could have picked any of the 26 identical tiles!

Rule 2: If something has a probability of 0, it is IMPOSSIBLE

Rule 3: All probabilities lie between 0 and 1, so if you find yourself with a negative answer, or something like 2.4, then you have done something wrong!!!

Question 4: What is the probability of picking the letter A, given that your friend tells you the tile in your hand is a vowel?

Now, believe it or not, this question is bordering on being A Level, but our good old formula still works!

There is only one letter A

Quick way of writing: "the probability of picking A"

$$P\,(A) = \frac{1}{5}$$

Seeing as our friend has told us that our tile is a vowel, there are 5 equally likely possibilities

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Big Example 2

Mr Barton is wondering what his Mum will have cooked him for tea. Going off past experience, the probability of it being beans on toast is 0.6, sausage and mash is 0.25, steak and chips is 0.1, and no food at all is 0.05

Question 1: What is the probability Mr Barton has beans on toast or sausage and mash ?

Now there is a key word in that question and it is OR. This means that Mr Barton can have either beans on toast or sausage and mash, it does not matter which occurs.

So what do you think we need to do with the probabilities?:

$$P(beans\ OR\ sausage) = P(beans) + P(sausage)$$
$$= 0.6 + 0.25 = 0.85$$

Now, events like this have a posh name – MUTUALLY EXCLUSIVE. All that means is that both events cannot occur at the same time – Mr Barton can't have both beans on toast and sausage and mash for tea… unless he is really hungry.

Rule 4: To find the probability of something happening OR something else happening, just add up your probabilities

Question 2: What is the probability Mr Barton actually gets his tea made?

Now one way to do this is to add up all the possible food outcomes… but there is a quicker way: There is only one outcome that results in no food, and the probability of any of the four outcomes occurring is 1 as it is certain that something will occur so:

$$P(food\ made) = 1 - P(no\ food)$$
$$= 1 - 0.05 = 0.95$$

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Question 3: What is the probability Mr Barton has beans on toast one night and the next?

Now, to answer this question, we must first make an assumption: what Mr Barton has for tea one night and the next night are INDEPENDENT of each other – in other words, the choice of last night's tea does not affect tonight's choice.

Now, that may seem pretty unrealistic, but you will tend to find that a lot of probability questions ask you to assume that two events are independent of each other.

Now, again there is a very important word in the question that helps you spot this type – AND.

To work out the probability of something happening and something else happening you do not add up the probabilities, as in this case you would get an answer bigger than 1, which is rubbish, so you do this...

$$P(\text{beans AND beans}) = P(\text{beans}) \times P(\text{beans})$$

$$= 0.6 \times 0.6 = 0.36$$

Rule 5: To find the probability of something happening AND something else happening, just multiply your probabilities together!

Classic Mistakes

The most common mistake pupils make with probability question is they mix up Mutually Exclusive and Independent events, and end up multiplying when they should be adding!

Learn to look for key words in questions:

Mutually Exclusive ⟶ Key Words: Or, Either ⟶ +

Independent ⟶ Key Words: And, Both, Together ⟶ ×

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What do you make of this argument...

If you toss two coins together, the probability of getting one head and one tail is: $\frac{1}{3}$ because...

$$P\ (head\ and\ tail) = \frac{1}{3}$$

One way you can get a head and a tail

Three equally likely outcomes: head-head, head-tail, or tail-tail

Sounds convincing, doesn't it?... Until you think about it and realise it's absolute rubbish!

1. There is not just "one way you can get a head and a tail"... there are two: head-tail and tail-head!

2. The three outcomes might be "equally likely", but there's one missing... tail-head!

So where did we go wrong?...

Well, when you have two experiments happening at the same time (like our two coins here), the safest way to ensure you account of all the outcomes is to knock up a SAMPLE SPACE DIAGRAM...

Coin 1

Coin 2

	Head	**Tail**
Head	H-H	H-T
Tail	T-H	T-T

So, we can now clearly see that there are 4 equally likely outcomes, and so get our probabilities, we just need to count up the number of outcomes we are interested in:

$$P\ (head\ and\ tail) = \frac{2}{4} = \frac{1}{2}$$

$$P\ (head\ and\ head) = \frac{1}{4}$$

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Big Example 3

I am feeling pretty bored so I decide to roll a pair of dice and each time subtract the highest score from the lowest

Question 1: Draw a sample space diagram to show all the equally likely outcomes.

Classic opportunity to use a sample space diagram – two experiments, each with lots of equally likely outcomes. Okay, so the outcomes from each dice go across the top and up the side, and the numbers in the middle come from subtracting the smallest number from the biggest.

Dice 1

Dice 2

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

Question 2: What is the probability of getting a score of 0?

36 equally likely outcomes, how many are 0?...

$$P\,(\textit{score of } 0) = \frac{6}{36} = \frac{1}{6}$$

Question 3: If you rolled the two dice 180 times, how many times would you expect to get a score of 1?

This is where we can use probabilities to help us predict results.
There are 36 equally likely outcomes, and 10 of them give us a score of 1.
So, if we rolled the dice 36 times, we'd expect to get a score of 1 on 10 occasions
So, What about if we rolled them 180 times?...

50 times!

$$\frac{10}{36} \underset{\times 5}{\overset{\times 5}{=}} \frac{50}{180}$$

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2. Tree Diagrams

What are Tree Diagrams, and when do you use them?

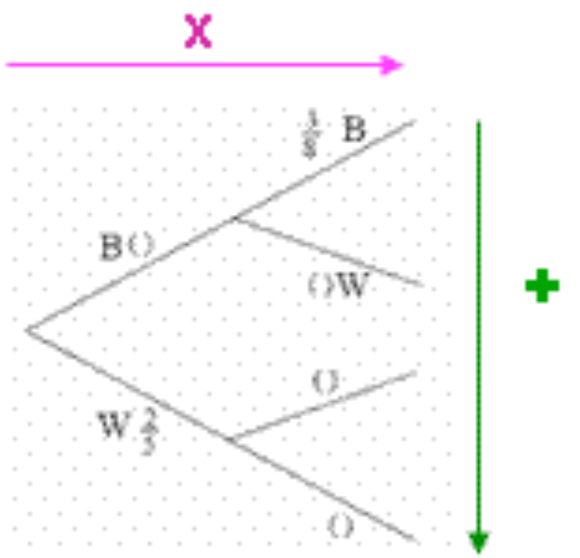
- Tree Diagrams are a very powerful tool in probability
- They are a very convenient way of representing a whole load of complicated information, which then allows you to answer some big mark questions without too much trouble.
- You tend to use tree diagrams to answer questions where there is more than one experiment going on at once, and the outcomes are not all equally likely

BONUS: Tree Diagrams can be used to answer questions involving both independent and non-independent events.

The Two Absolutely Crucial Rules of Tree Diagrams

1. We MULTIPLY probabilities going ACROSS

2. We ADD probabilities going DOWN



NOTE: And a really good way to check you have down everything right is to add up all the probabilities are the end of your branches... because you know that the sum of the probabilities of all outcomes must add up to 1!

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Example 1

Sarah is bored – very bored - so she puts twelve coloured cubes in a bag. Five of the cubes are red and 7 are blue. She decides it would be fun to remove a cube at random from the bag and note the colour before replacing it. For even more fun she then chooses a second cube at random. What is the probability she pulls out two beads of the same colour?

Okay, so what are the things we should be thinking about when we knock up a tree diagram?...

1. What are our two experiments so we can spilt up our tree diagram?...

Well, what about Sarah's "first pick" and then "second pick"?

2. Do we know what the probability of picking a red cube is?...

Well, there are 12 cubes in the bag, and 5 of them are red, so...

$$P(red) = \frac{5}{12}$$

3. How about a blue cube?...

Again, 7 blues out of the 12 in the bag, so...

$$P(blue) = \frac{7}{12}$$

4. On our second pick, do our probabilities change?...

Well, there is a crucial little phrase hidden in the question: "replacing it". Because Sarah pops the bead back into the bag after each pick, whatever she gets on her first pick has no effect whatsoever on what she gets on here second, so the probabilities remain the same.

If you want to be really fancy about this (and why not!), you could say that because Sarah replaces the cubes, the events are INDEPENDENT of each other!

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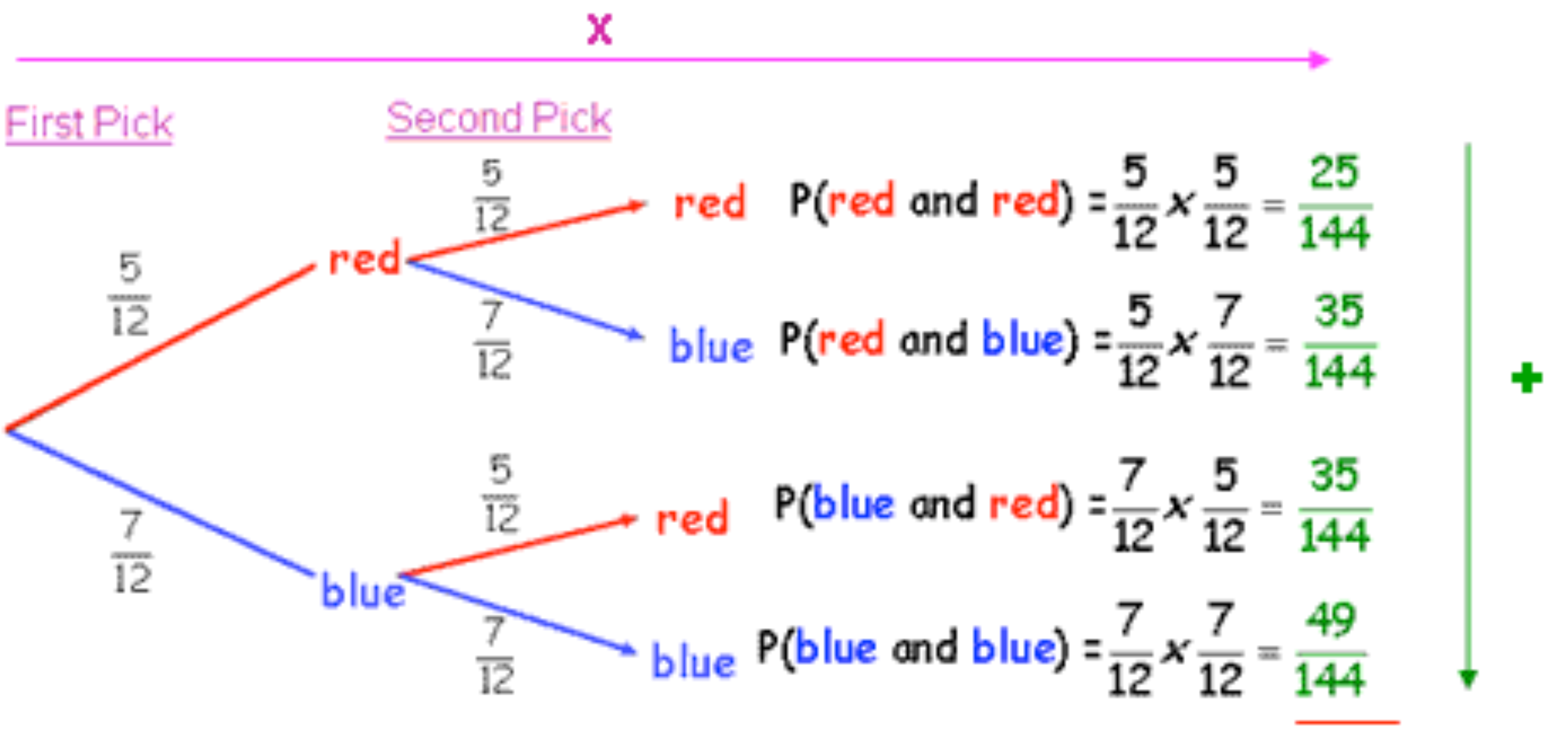

P(red and red) = $\frac{5}{12} \times \frac{5}{12} = \frac{25}{144}$

P(red and blue) = $\frac{5}{12} \times \frac{7}{12} = \frac{35}{144}$

P(blue and red) = $\frac{7}{12} \times \frac{5}{12} = \frac{35}{144}$

P(blue and blue) = $\frac{7}{12} \times \frac{7}{12} = \frac{49}{144}$

$\frac{144}{144} = 1$

Note: I wouldn't bother simplifying my fractions here, as it makes the adding up easier when they have the same denominator!

Check: Our probabilities add up to 1, so it's looking good!

Question: What is the probability she gets two beads of the same colour?...

Well, the end of which braches give us that?

$P(same\ colour)$

$= P(red\ and\ red) + P(blue\ and\ blue)$

$= \frac{25}{144} + \frac{49}{144} = \frac{74}{144}$ Now we can simplify $\longrightarrow \frac{37}{72}$

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Example 2

For many years, Hannah and George have been locked in some pretty heated games of Scrabble and Monopoly. The probability that Hannah wins at Scrabble is 0.7, and the probability that George wins at Monopoly is 0.65. One rainy day they sit down for another fierce battle. What is the probability George wins both games?

Okay, before we start, let's make sure we know what's going on here...

1. What are our two experiments so we can spilt up our tree diagram?...

Well, what about "Scrabble" and then "Monopoly"?

2. We know the probability Hannah wins at Scrabble is 0.7, but what about George?...

Well, either one wins, or the other (we assume no draws), so the two probabilities must add up to 1

So...

$$P(G \text{ wins Scrabble}) = 1 - P(H \text{ wins Scrabble}) = 1 - 0.7 = 0.3$$

3. And how about Hannah winning at Monopoly?...

It's a similar sort of thing...

$$P(H \text{ wins Mon}) = 1 - P(G \text{ wins Mon}) = 1 - 0.65 = 0.35$$

4. On the second game, do our probabilities change?...

Well, because the question does not say so, we must assume that the probabilities stay the same, and the results in Scrabble and Monopoly are INDEPENDENT.

You might argue that if Hannah wins at Scrabble, then George will be more determined to stuff her friend at Monopoly, but the question is trying to make life easy for us, so let's let it!

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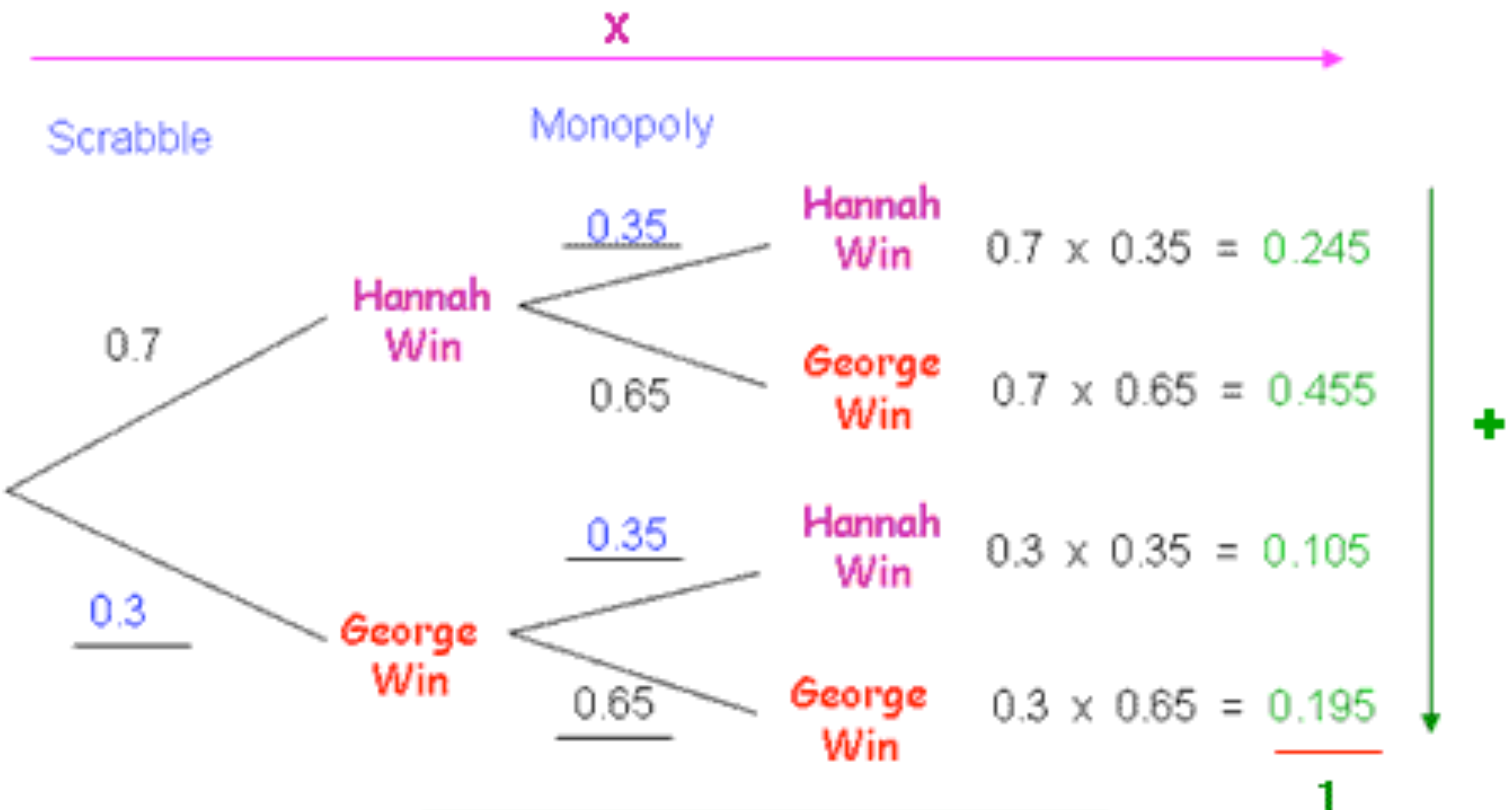


x

Scrabble

Monopoly

0.7

Hannah Win

0.35

Hannah Win

0.7 x 0.35 = 0.245

0.65

George Win

0.7 x 0.65 = 0.455

0.3

George Win

0.35

Hannah Win

0.3 x 0.35 = 0.105

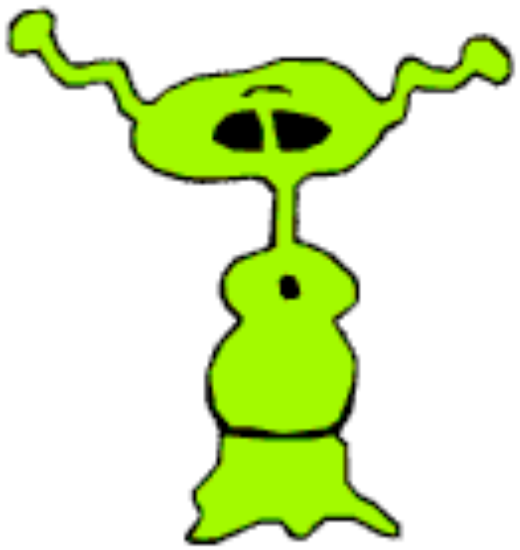
0.65

George Win

0.3 x 0.65 = 0.195

+

1

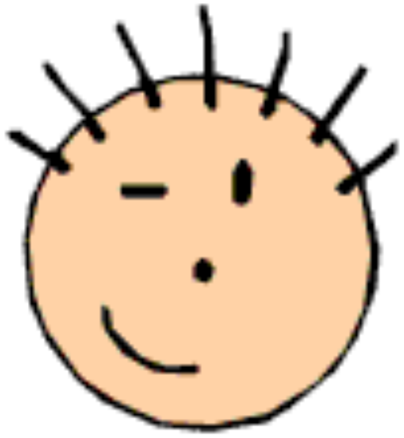


Check: Our probabilities add up to 1, so it's looking good!

Question: What is the probability George wins both games?

Well, we just follow the bottom branch...

$$P(George\ wins\ both) = 0.3 \times 0.65 = 0.195$$



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Example 3

Sarah is bored again, so it's back to the bag of beads! However, this time she really decides to spice things up. She still has 12 beads, but this time there are 5 red, 6 blue and 1 green. Crazier still, when she picks one out this time, she decides not to put it back! What is the probability that after two picks, Sarah has two beads that are the same colour?

Okay, this is a bit of a tricky one, so let's try and get our heads around what is going on...

1. What are our two experiments so we can spilt up our tree diagram?...

Well, I reckon it must be "first pick" and then "second pick"?

2. The probabilities on the first pick should be easy enough: $P(red) = \frac{5}{12}$ $P(blue) = \frac{6}{12}$ $P(green) = \frac{1}{12}$

3. It's on the second pick that things start getting tricky.

Say Sarah picks a red out first, what is the probability of her picking a red out second?...

Well, there are now only 4 reds in the bag, and there are only 11 beads as well! $P(red) = \frac{4}{11}$

Whenever things are not replaced, you have to think very carefully about the probabilities on your branches!

4. On the second pick, do our probabilities change?...

In short, no they don't! Again, there is a crucial phrase: "not to put it back". This means that whatever happens on the first pick DOES affect the probabilities on the second pick, so these events are... DEPENDENT!

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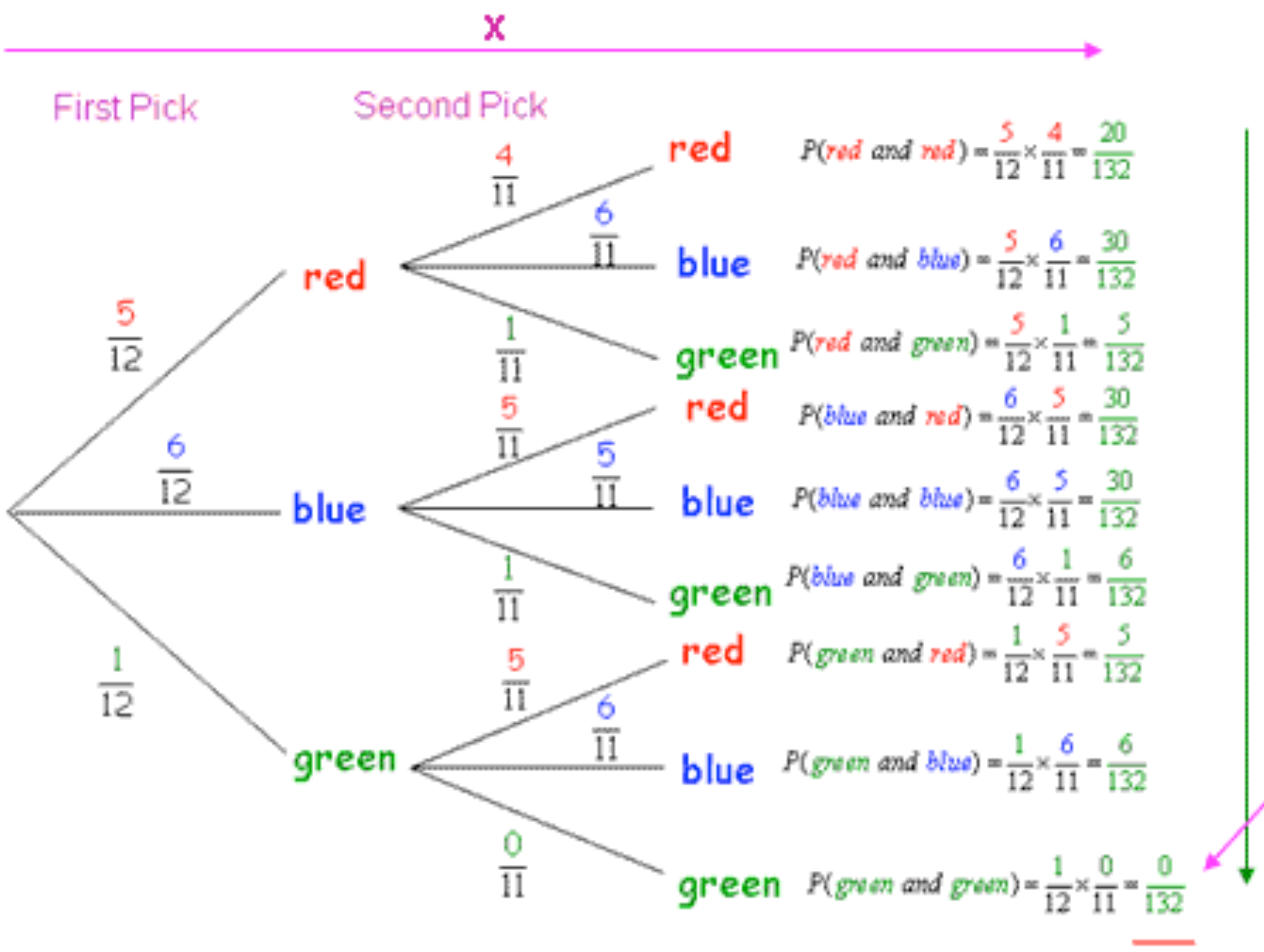

$$P(red\ and\ red)=\frac{5}{12}\times\frac{4}{11}=\frac{20}{132}$$

$$P(red\ and\ blue)=\frac{5}{12}\times\frac{6}{11}=\frac{30}{132}$$

$$P(red\ and\ green)=\frac{5}{12}\times\frac{1}{11}=\frac{5}{132}$$

$$P(blue\ and\ red)=\frac{6}{12}\times\frac{5}{11}=\frac{30}{132}$$

$$P(blue\ and\ blue)=\frac{6}{12}\times\frac{5}{11}=\frac{30}{132}$$

$$P(blue\ and\ green)=\frac{6}{12}\times\frac{1}{11}=\frac{6}{132}$$

$$P(green\ and\ red)=\frac{1}{12}\times\frac{5}{11}=\frac{5}{132}$$

$$P(green\ and\ blue)=\frac{1}{12}\times\frac{6}{11}=\frac{6}{132}$$

$$P(green\ and\ green)=\frac{1}{12}\times\frac{0}{11}=\frac{0}{132}$$

$$\frac{132}{132}=1$$

Note: a probability of zero here makes sense as there is only 1 green!

Check: Our probabilities add up to 1, so it's looking good!

Question: What is the probability she gets two beads of the same colour?...

$$P(same\ colour)=P(red\ and\ red)+P(blue\ and\ blue)+P(green\ and\ green)$$

$$=\frac{20}{132}+\frac{30}{132}+0=\frac{50}{132}=\frac{25}{66}$$

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Thinking like a Tree Diagram

Sometimes you can answer a question by picturing a tree diagram in your head and imagining the branches, without actually drawing one. This might just save you some precious minutes in an exam...

Example 4

The probability I somehow find the energy to go to the gym on Monday is 0.3. If a miracle happens and I do go to the gym on Monday, the probability I go again on Tuesday falls to 0.1. If I don't go on Monday, the probability remains the same. What is the probability that:

(a) I go to the gym on both days

(b) I go to the gym on just one day?

(a) Right, let's think about this... I need to go on both days... well the probability I go to the gym on Monday is 0.3... and if I go on Monday, the probability I also go on Tuesday falls to 0.1... so to find the probability I go on both days I would travel along both branches of my tree diagram, so I must MULTIPLY!

$$P(\text{gym Mon and Tues}) = 0.3 \times 0.1 = 0.03$$

(b) This is a bit trickier... I only go to the gym on one day... how could that happen?... Well, I could go on Monday, and then not go on Tuesday... OR I could give Monday a miss, and then go Tuesday!... So what would be the probabilities of those?...

$$P(\text{gym Mon but not Tues}) = 0.3 \times 0.9 = 0.27$$

$$P(\text{no gym Mon but go Tues}) = 0.7 \times 0.3 = 0.21$$

And these would be the ends of the branches, so to get the probability of either happening, I need to ADD: → 0.48

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3. Averages and Measures of Spread

What are Averages and Measures of Spread, and why do we need them?

- Averages and Measures of Spread are two of the most important and useful concepts in maths
- They allow us to look at a huge load of data and make some sense out of it, summarise it, and compare it to other huge sets of data.
- Averages and Measures of Spread are used every single day, whether it be crowd attendance at Old Trafford, viewing figures for Lost, or the salary of your average poor maths teacher.

The usefulness of the different types of Averages and Measures of Spread will be looked at in The Big Cricket Example, so for now, let's learn how to work them out!

1. The Mean

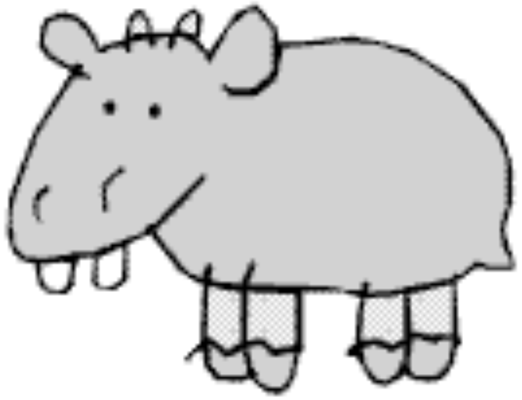
Whenever most people talk about an average, this is the one they... mean!

How to work out the Mean:

1. Add up all your data values
2. Divide this total by the number of data values

What about a little rhyme?:

I'm not really sure
What the MEAN is about
Just add them all up
And share them all out



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2. The Median

This is the one people tend to mess up!... Don't let it happen to you!

How to work out the Median:

1. Place all your data values in ascending order (biggest to smallest)
2. The piece of data in the middle is your median

NOTE: If you have an EVEN number of data values, there will be TWO pieces of data in the middle. No problem, just add them together and divide by two to find the number halfway between them... and this is your median!

What about a little rhyme?:

I don't know the rhyme
I don't know the riddle
I am the MEDIAN
And I'm in the middle!

3. The Mode

This is the final type of average, and the easiest one to work out... so long as you remember how!

How to work out the Mode:

1. Find the most common piece of data (number or letter) and this is your mode!

NOTE: You can have no modes or more than one mode, and you must write them all down!

What about a little rhyme?:

I don't want to brag
I don't want to boast
I am the MODE
I am the most

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4. The Range

The Range is a Measure of Spread, and tells you... well, how spread out the data is!

How to work out the Range:

1. Subtract the smallest data value away from the biggest data value!

What about a little rhyme?:

From largest to smallest
See how they change
Take them away
And I am the RANGE

NOTE: It's one thing knowing how to work out the three averages and the measure of spread, but it's just as important to know how to interpret them! Hopefully this example will help!

The Big Cricket Example

Andrew Flintoff and Michael Vaughn are having an argument in the pub trying to decide who has had the better season with the cricket bat. Here are their scores:

20	35	22	55	60
10	17	32	64	86
14	32	50	24	30

0	0	0	2	0
15	5	370	250	
0	3	5	0	1

Use your knowledge of Averages and Measures of Spread to decide which cricketer has had the better season

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NOTE: The first point to notice is that by just looking at the scores as they are makes it hard to come to a decision about who has had the better season... that's why we need Statistics!

Secondly, just adding up the total amount of runs and deciding that way would not be fair. Why?... well, because they have not played the same number of games!

So, there is only one thing for it... let's work out some statistics!

1. The Mean

1. Add up all your data values
2. Divide this total by the number of data values

Andrew Flintoff

total runs scored: 551

total games played: 15

mean: 36.7 runs (1dp)

Michael Vaughn

total runs scored: 651

total games played: 14

mean: 46.5 runs

What does this tell us? – well, it looks like, on average, Michael Vaughn has had the better season

Good thing about the mean – notice how every single score was used to calculate the mean – this means it gives a good summary of the whole season.

Bad thing about the mean – look at Michael Vaughn's scores. He only had two decent ones, and yet his mean is far higher than Andrew Flintoff's! This is because the mean is significantly affected by outliers – pieces of data which stand out for being really low or really high like the two scores of 370 and 250. You could argue that these have distorted the result!

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2. The Median

1. Place all your data values in ascending order (biggest to smallest)
2. The piece of data in the middle is your median

$$10\quad 14\quad 17\quad 20\quad 22\quad 24\quad 30\quad \boxed{32}\quad 32\quad 35\quad 50\quad 55\quad 60\quad 64\quad 86$$

Median = 32 runs

Note: there are the same number of data (7 pieces) either side of the box!

$$0\quad 0\quad 0\quad 0\quad 0\quad 0\quad \boxed{1\quad 2}\quad 3\quad 5\quad 5\quad 15\quad 250\quad 370$$

$$(1+2) \div 2 = 1.5$$

Median = 1.5 runs

Note: there are still the same number of data (6 pieces) either side of the box!

What does this tell us? – well, this time it looks like Andrew Flintoff has had the better season

Good thing about the median – because we are only focussing on pieces of data in the middle, outliers don't have as big an effect, so they cannot distort the results!

Bad thing about the median – the problem here is that you are only looking at – at most – a couple of pieces of data from each player. You could argue that the result is not representative as a lot of pieces of data (scores) are just ignored!

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3. The Mode

1. Find the most common piece of data (number or letter) and this is your mode!

Andrew Flintoff

mode: 32 runs

Michael Vaughn

mode: 0 runs

What does this tell us? – well, using the mode it again looks like Andrew Flintoff is on top!

Good thing about the mode – very speedy to work out!

Bad thing about the mode – can give distorted, or even no results. Imagine if Andrew Flintoff only scored one innings of 32 runs… he would have no mode to compare! Or, imagine if he instead scored a couple of innings of 200… the mode would then say this was his average!

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4. The Range

1. Subtract the smallest data value away from the biggest data value!

Andrew Flintoff

largest value: 86

smallest value: 10

range: 76 runs

Michael Vaughn

largest value: 370

smallest value: 0

range: 370 runs

What does this tell us? - well, one answer that I often here is this: "Michael Vaughn has the biggest range, so he is the best!"... but that's not quite right.

The bigger the range, the more spread out your scores are... so the less consistent (brilliant maths word that always impresses examiners/teacher) your performance is.

So, I would argue, because Andrew Flintoff has a smaller range, his performance is more consistent, and therefore he has had the better season!

Good thing about the range - gives a very quick measure of how spread out the data is

Bad thing about the range - unfortunately, this statistic is vulnerable to outliers as well! Michael Vaughn had a couple of big scores, and look at the effect it had on his range!... This is why mathematicians prefer to measure the spread of data using the Inter-quartile Range or Standard Deviation... but don't worry about them yet!

So who is the better cricketer?...

Well, in the end, it's up to you! The most important thing is that you have shown you can calculate each of the statistics and - not a lot of people can do this - interpret what they mean!

If you want my opinion, as a proud Lancastrian, Andrew Flintoff is much better!

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Estimating the Mean from Grouped Data

Example:
Calculate an estimate of the mean number of fans attending the mighty Preston North End football matches from the following table:

Attendance	Frequency
$0 < A \le 5{,}000$	5
$5{,}000 < A \le 10{,}000$	12
$10{,}000 < A \le 15{,}000$	24
$15{,}000 < A \le 20{,}000$	8

Okay, now do you see a problem here?... Look at the first group... we know there were 5 matches where between 0 and 5,000 people turned up, but we don't know exactly how many people were at those matches!... One match could have had 1,309... another 4,510... we just don't know!

So... the best we can do is to make an estimate!

And what is our best estimate for that first group?... Well, the MID-POINT... 2,500!

And that is how we calculate an estimate for the mean from grouped data:

1. Work out the mid-point

2. Work out the mid-point x Freq for each group

3. Use this formula:

$$\text{Mean} = \frac{\text{Sum of Mid-Point x Freq}}{\text{Total Frequency}}$$

Attendance	Mid-Point	Frequency	Mid-Point x Freq
$0 < A \le 5{,}000$	2,500	5	12,500
$5{,}000 < A \le 10{,}000$	7,500	12	90,000
$10{,}000 < A \le 15{,}000$	12,500	24	300,000
$15{,}000 < A \le 20{,}000$	17,500	8	140,000
TOTALS		**49**	**542,500**

$$\text{Mean} = \frac{542500}{49}$$

$= 11{,}071$ (nearest whole)

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4. Cumulative Frequency and Box Plots

Why do we bother with Statistical Diagrams?

- The answer to this question is similar to the one for: "why do we bother working out averages and measures of spreads?".

- We live in a world jam-packed full of statistics, and if we were forced to look at all the facts and figures in their raw, untreated form, not only would we probably not be able to make any sense out of them, but there is also a very good chance our heads would explode.

- Statistical Diagrams – if they are done properly - present those figures in a clear, concise, visually pleasing way, allowing us to make some sense out of the figures, summarise them, and compare them to other sets of data.

1. What is Cumulative Frequency?

Cumulative is just a posh way of saying "add up as you go along"

Frequency is just a posh word for "total"

So... if you put them together, you get a very posh way of saying "add the totals up as you go along"

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Big Example

To the right is a table showing the length of time a group of 40 Year 10 students spent playing on the Nintendo Wii on a gloomy week in January. Draw a Cumulative Frequency Curve, use it to estimate the Median and Inter-Quartile Range, and construct a Box Plot

Hours spent playing	Frequency
0 < h ≤ 1	2
1 < h ≤ 2	5
2 < h ≤ 3	10
3 < h ≤ 4	15
4 < h ≤ 6	5
6 < h ≤ 10	3

2. Adding a Cumulative Frequency Column

Before you can even start thinking about drawing a Cumulative Frequency Curve, you need to be able to add a Cumulative Frequency column to your Frequency table.

Remember, Cumulative Frequency just means that you add up the frequencies as you go along, so that is exactly what you do!

Hours spent playing	Frequency	Cumulative Freq
0 < h ≤ 1	2	2
1 < h ≤ 2	5	7
2 < h ≤ 3	10	17
3 < h ≤ 4	15	32
4 < h ≤ 6	5	37
6 < h ≤ 10	3	40

This is the number of people who play for 1 hour or less

This is the number of people who play for 2 hours or less (5 + 2)

This is the number of people who play for 3 hours or less (5 + 2 + 10)

Check: This final entry should always equal the total frequency!

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3. Drawing the Cumulative Frequency Curve

Remember: we plot Cumulative Frequency (y axis) against the upper boundary of each group (x axis)

So… for group one it's 1 on the x axis and 2 on the y

and for group two, it's 2 on the x axis and 7 on the y…

Hours spent playing	Frequency	Cumulative Freq
$0 < h \le \boxed{1}$	2	$\boxed{2}$
$1 < h \le \boxed{2}$	5	$\boxed{7}$
$2 < h \le 3$	10	17
$3 < h \le 4$	15	32
$4 < h \le 6$	5	37
$6 < h \le 10$	3	40

Things to notice about the Cumulative Frequency Curve:

1. When you have finished plotting the points, join them up with a smooth curve.

2. Native the curve starts at (0, 0). This is because there is nobody playing less than 0 hours a week!

3. You must label your axis correctly, or you lose very easy marks!

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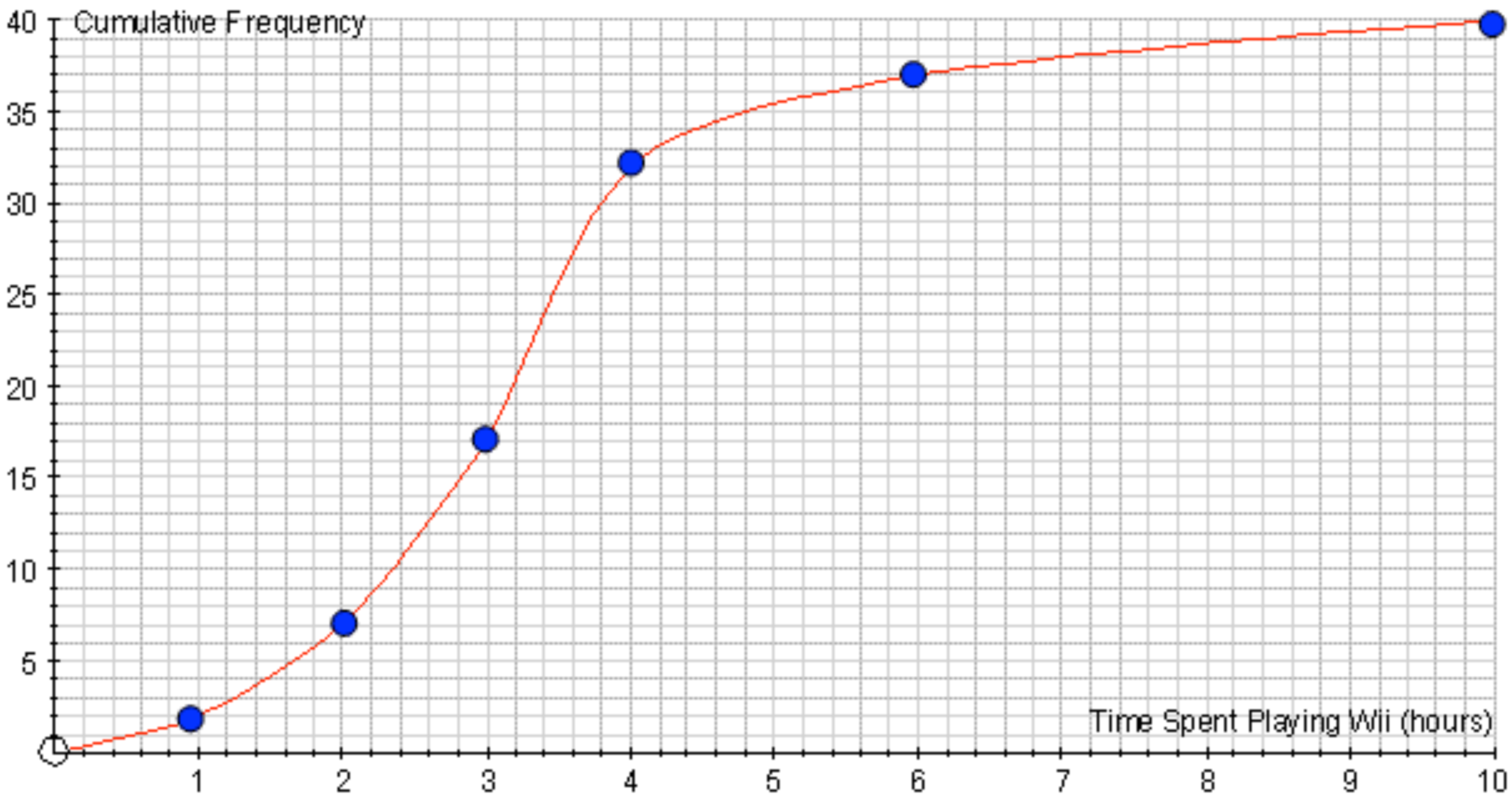

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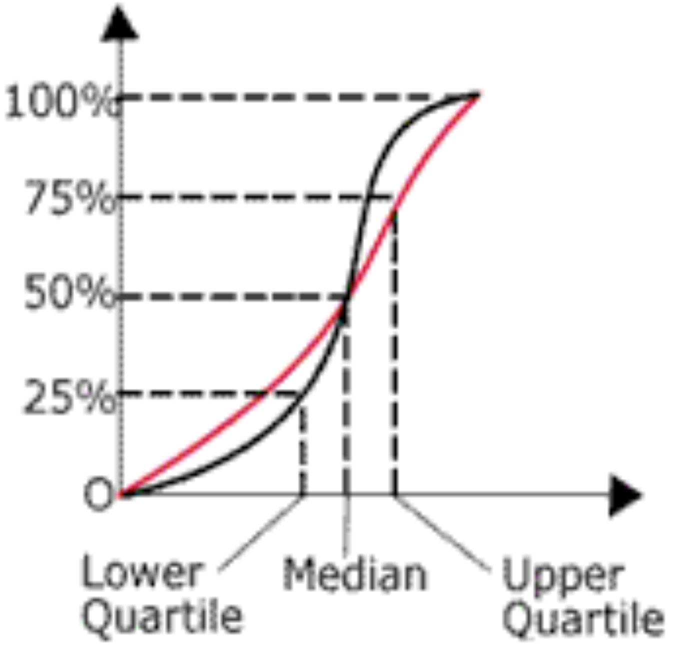
4. Estimating the Median and Inter-Quartile Range

We have spent a while drawing our cumulative frequency curve, so we may as well use it. Very quickly we can come up with estimates for the Median and the Inter-Quartile Range

(a) Median

As you hopefully remember, the Median is the MIDDLE value. To find it we:

1. Work out what is 50% of our total frequency (half way up the y axis)
2. Draw a horizontal line across until it hits our curve
3. When it hits the curve, draw a vertical line down to the x axis
4. The value on the x axis is our Median



(b) Inter-Quartile Range

For this we need to work out the upper quartile (UQ) and the lower quartile (LQ), and then calculate: UQ - LQ

To find the Upper Quartile:

1. Work out what is 75% of our total frequency (three-quarters of the way up the y axis)
2. Draw a horizontal line across until it hits our curve
3. When it hits the curve, draw a vertical line down to the x axis
4. The value on the x axis is our Upper Quartile

The Lower Quartile is the same, but 25% (one-quarter) of the way up!

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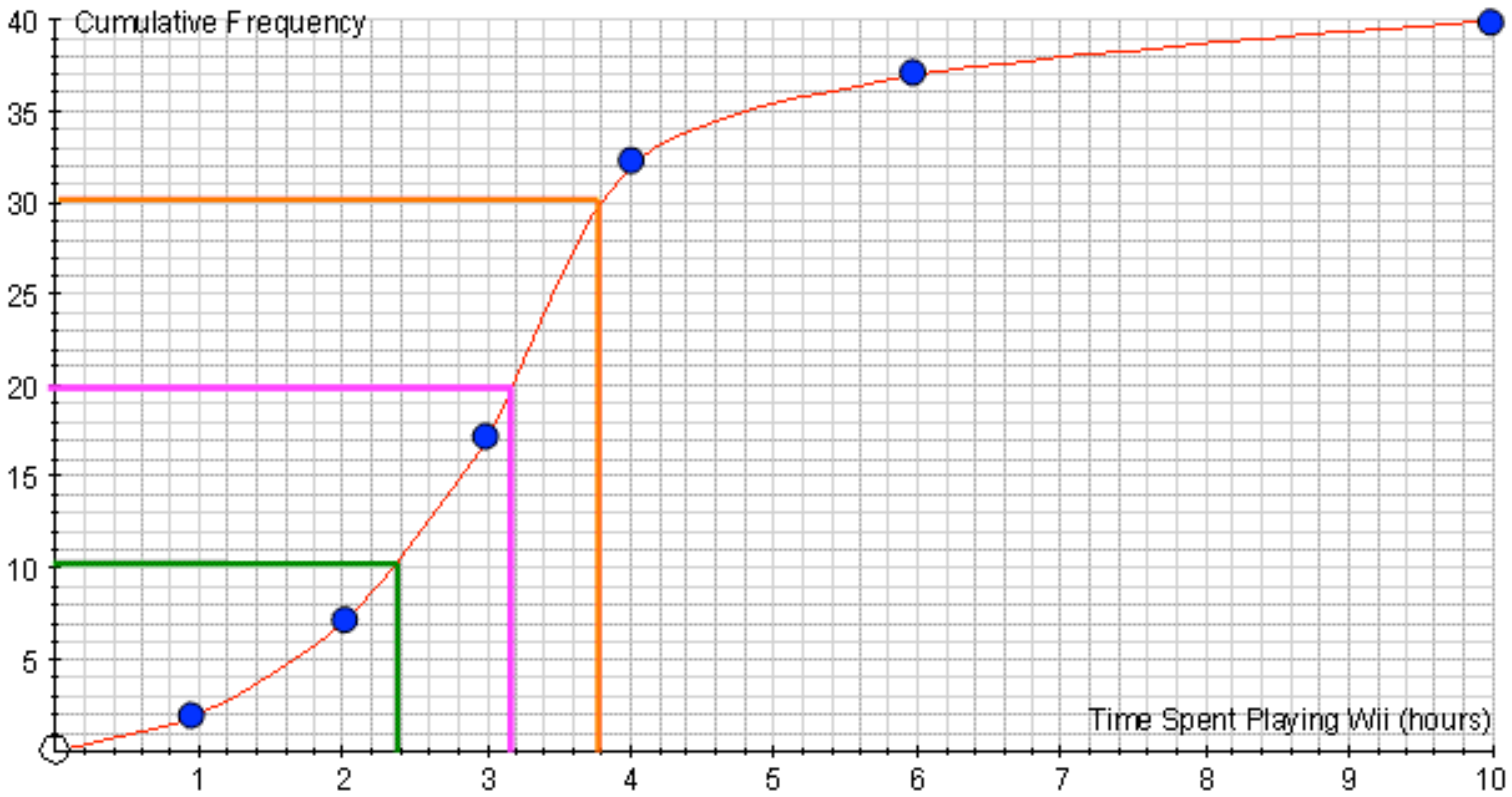

Median:

50% of 40 = 20

Median = 3.2 hours

Upper Quartile

75% of 40 = 30

UQ = 3.8 hours

Lower Quartile

25% of 40 = 10

LQ = 2.4 hours

Inter-Quartile Range

= UQ - LQ

= 3.8 - 2.4

= 1.4 hours

Remember: The Median is a form of average, and just like the Range, The Inter-Quartile Range is a measure of consistency

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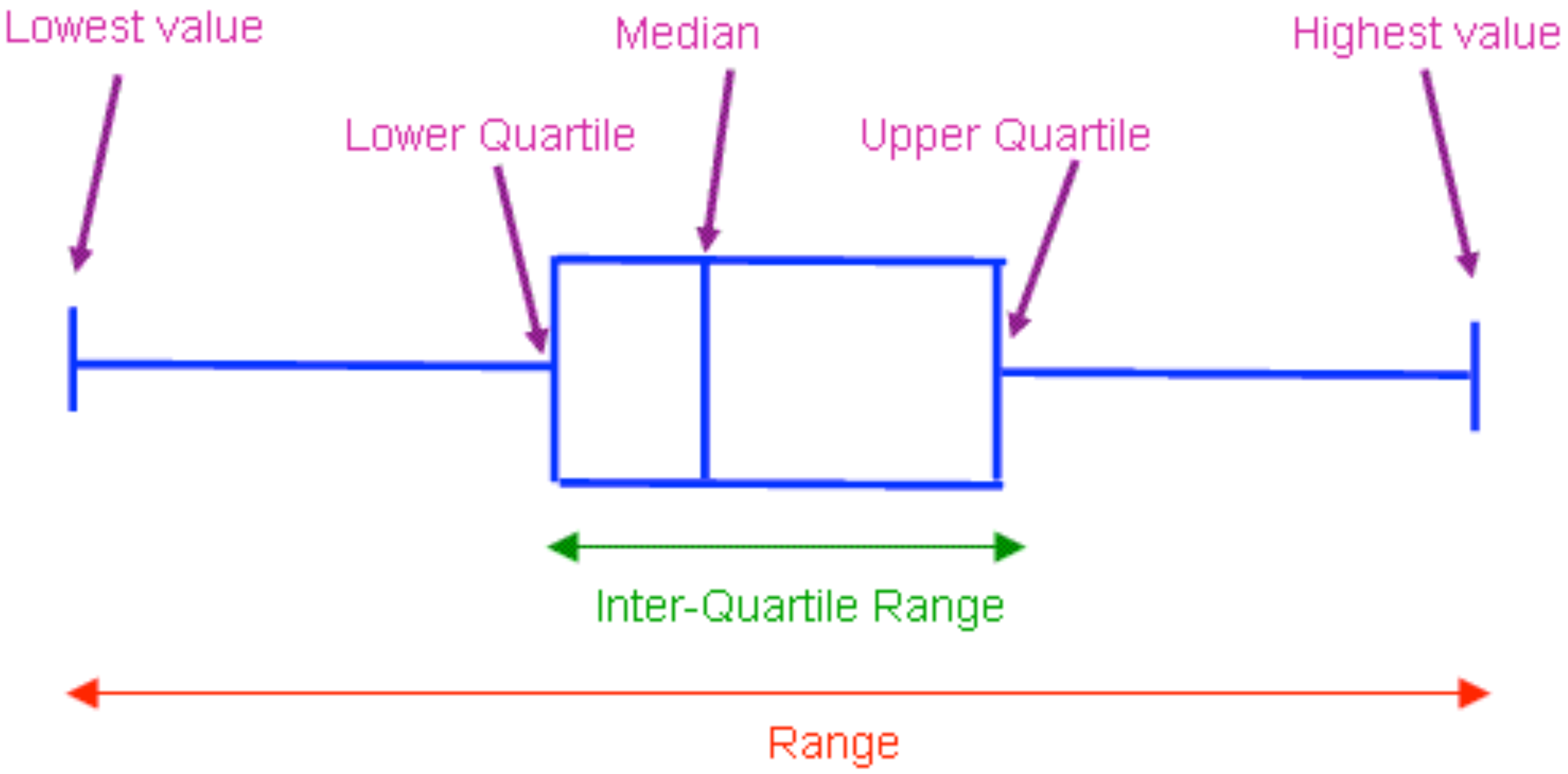
5. Drawing Box Plots

Box Plots are another way of representing all the same information that can be found on a Cumulative Frequency graph.

Top Tip: if you have the chance, draw your box plot directly below your cumulative frequency graph, using the same scale on the x axis, and you can just extend the vertical lines downwards and save yourself a lot of time!



Note: The minimum value is the lowest possible value of your first group, and the maximum value is the highest possible value of your last group

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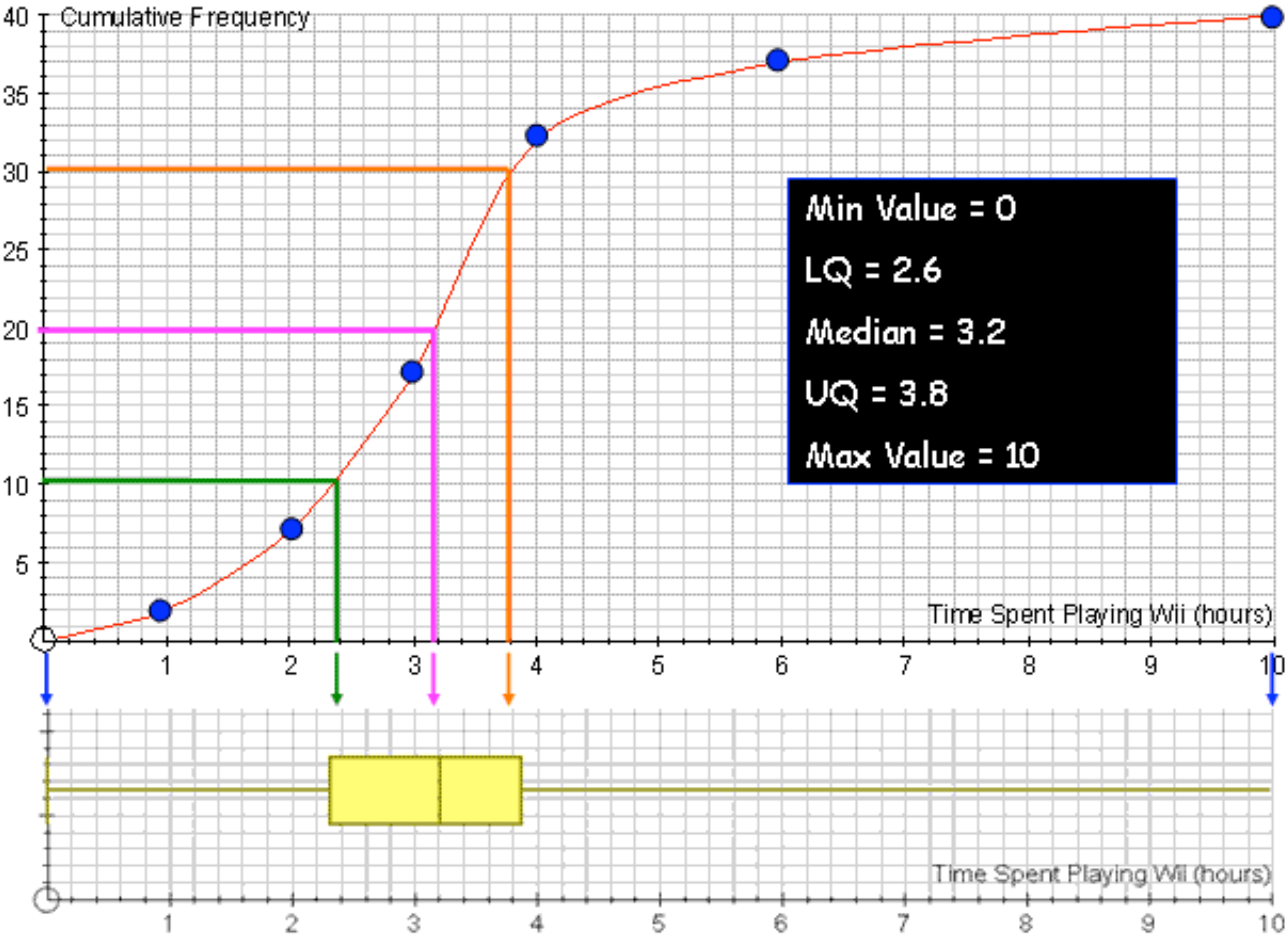

Min Value = 0

LQ = 2.6

Median = 3.2

UQ = 3.8

Max Value = 10

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5. Pie Charts

Why do we bother with Statistical Diagrams?

- The answer to this question is similar to the one for: "why do we bother working out averages and measures of spread?".

- We live in a world jam-packed full of statistics, and if we were forced to look at all the facts and figures in their raw, untreated form, not only would we probably not be able to make any sense out of them, but there is also a very good chance our heads would explode.

- Statistical Diagrams - if they are done properly - present those figures in a clear, concise, visually pleasing way, allowing us to make some sense out of the figures, summarise them, and compare them to other sets of data.

Big Example

A group of 72 maths teachers were asked to choose their favourite TV show from a list, and their responses are shown in the table on the right. Construct a pie chart to illustrate this information

TV Show	Total
Lost	12
Heroes	10
Desperate Housewives	4
Countdown	15
Teachers TV	13
The Beauty of Maths	18

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1. Working out the Angles

- Before you can start to draw the pie chart, you need to know how big a slice each of the choices is going to take up – in other words, you need to know the angle of each segment
- To work this out, you need to remember that there are 360 degrees in a circle
- That means there are 360 degrees to share between each of the people who took part in the survey
- How many degrees does each person get?... Well, divide 360 by the number of people surveyed!

To Calculate the Angles

1. Add up the total number of pieces of data
2. Divide 360 by this number – this tells you how many degrees is allocated to each piece of data
3. To work out the size of angle for each category, multiply the answer to **2.** by the number of people in each category – rounding your answers sensibly if you need to.
4. Check: Before you start to draw, make sure you check that your total number of degrees does add up to 360!

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Our Example:

1. So, we have a total of 72 teachers who were surveyed.

2. 360 ÷ 72 = 5

So... each teacher is worth 5 degrees on our pie chart

3. We know how many teachers are in each segment, so let's use our answer to **2.** to work out what angle each segment gets

TV Show	Total	Working Out	Angle of Segment
Lost	12	12 × 5 = 60	60°
Heroes	10	10 × 5 = 50	50°
Desperate Housewives	4	4 × 5 = 20	20°
Countdown	15	15 × 5 = 75	75°
Teachers TV	13	13 × 5 = 65	65°
The Beauty of Maths	18	18 × 5 = 90	90°

Remember: Check this column adds up to 360 before you move on!

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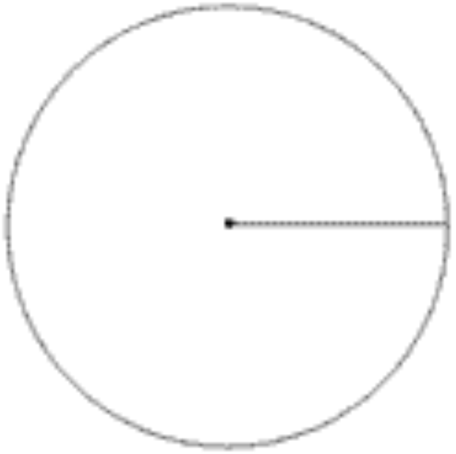
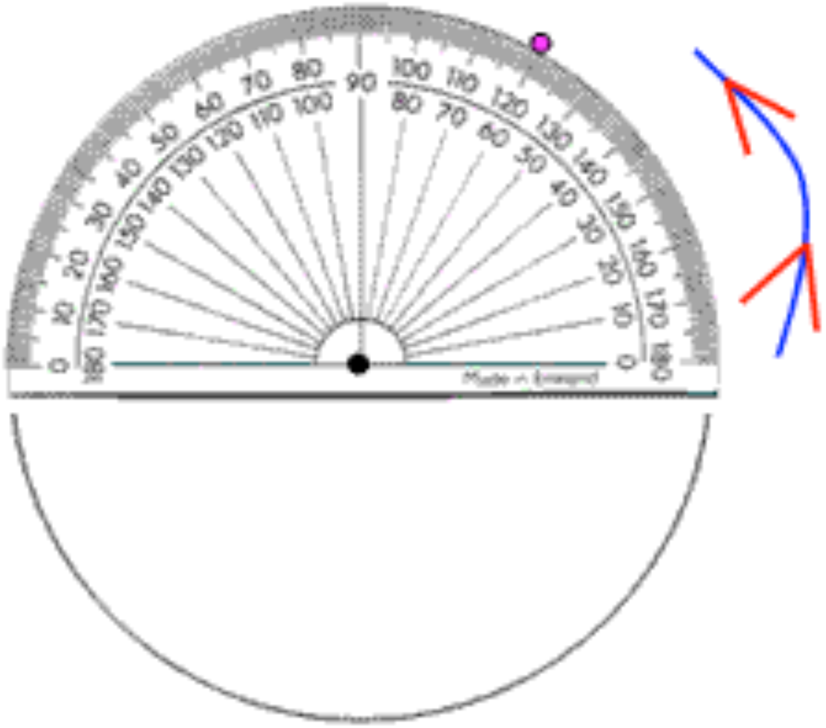
2. Drawing the Pie Chart

You've done all the hard work, and drawing the pie chart should be easy… but you'll be amazed how many people mess it up, so take your time and follow these steps…

1. Draw a circle using a compass. Mark the centre with a dot and draw a straight line from the centre up to the right of your circle



2. Carefully place your angle measurer along the line, with the centre exactly on the centre of the circle. Now, count around from 0 until you reach the correct number of degrees – in this case 60^0 – and place a dot



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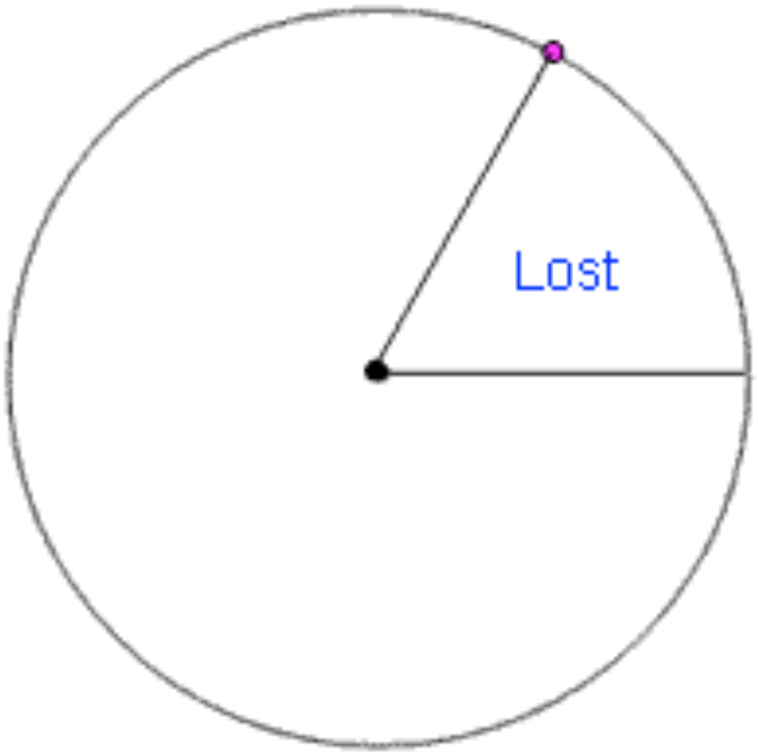
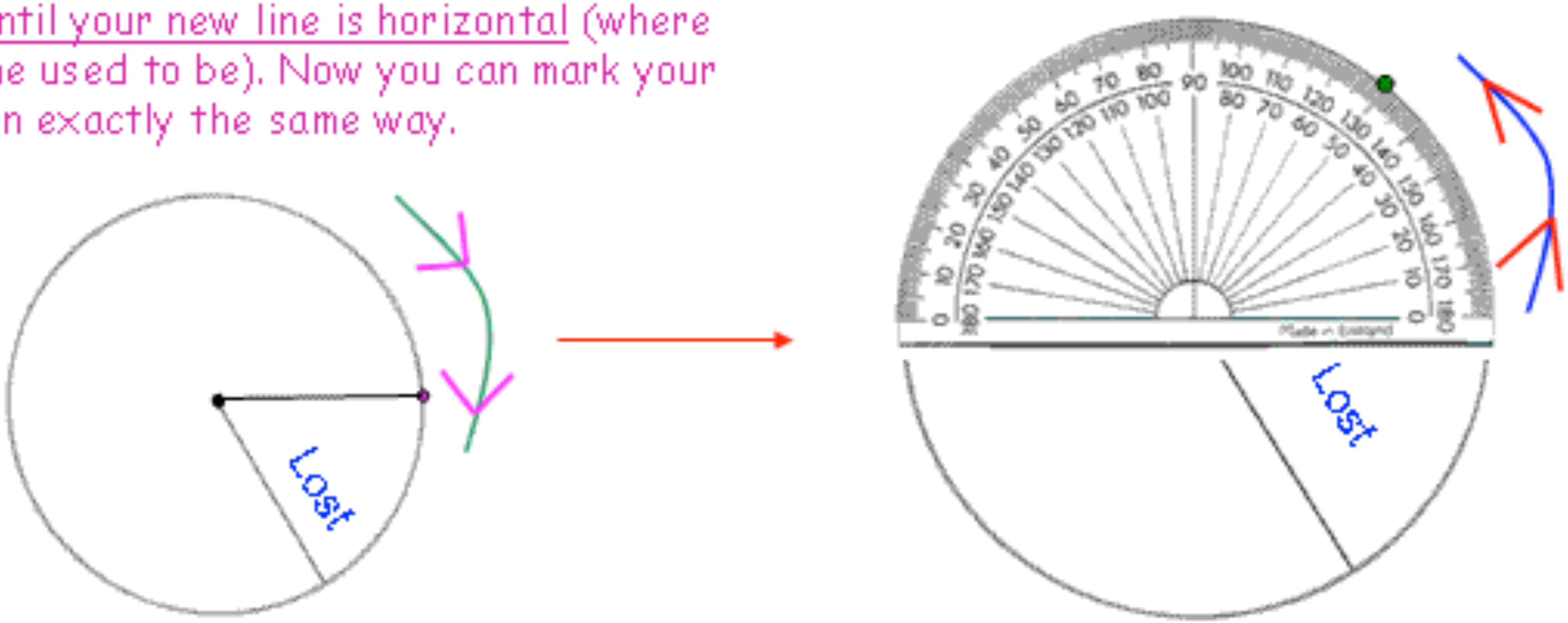
3. Join up your dot to the centre with a straight line, and label your segment.



4. Now, this is the tricky bit... turn your pie chart clockwise until your new line is horizontal (where the first line used to be). Now you can mark your next angle in exactly the same way.



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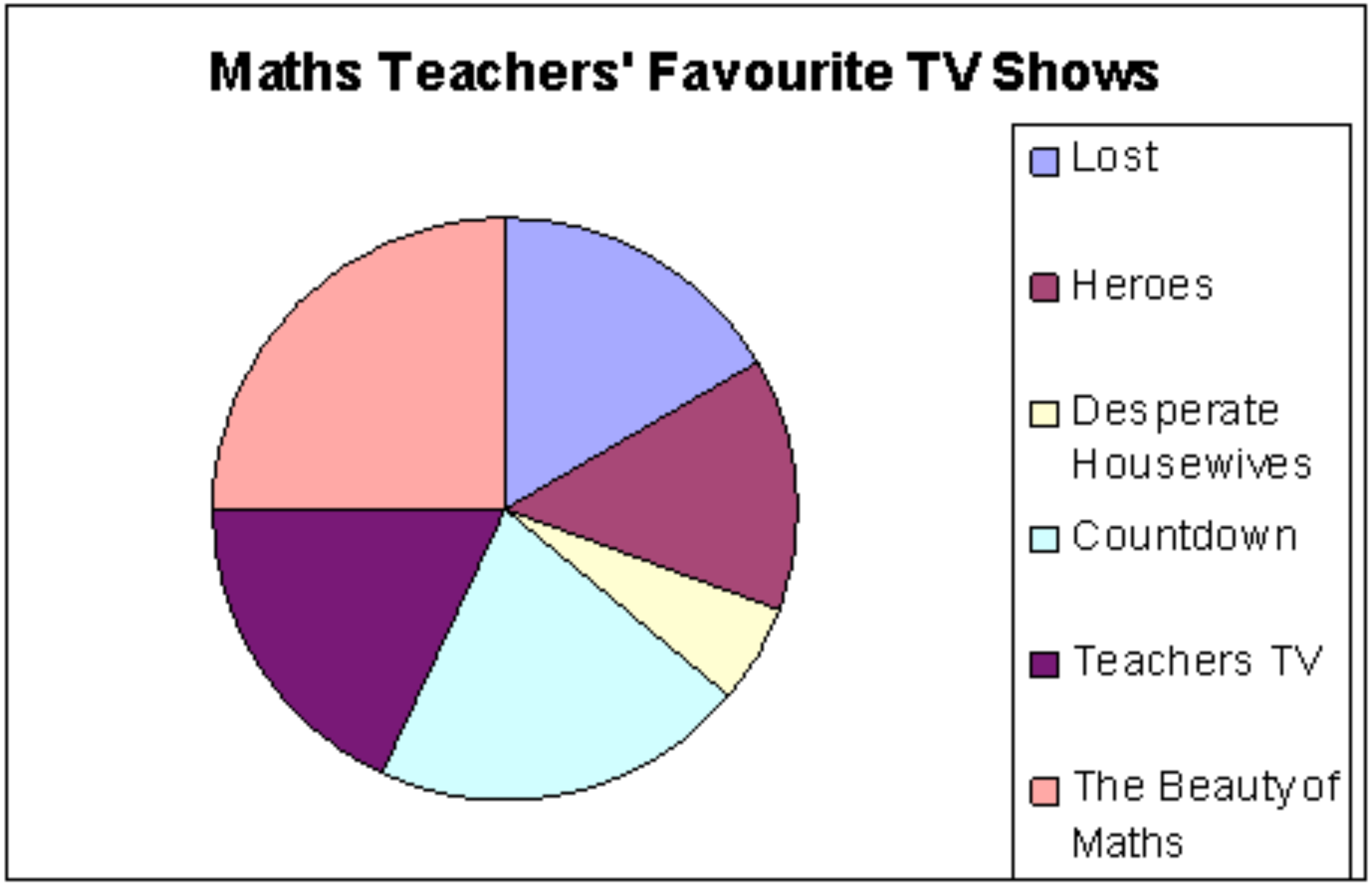
5. Keep doing this until you have drawn all your segments

Check: You will know if you have got it right if the line to make your final segment is the very first line you drew!

6. If you want to you can colour in your segments, but you must remember to label them clearly, or add a key!



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3. What CAN we tell from Pie Charts

- Well, if you look back at our pie chart, you will see that it shows pretty clearly that The Beauty of Maths was the most popular choice amongst our maths teachers, whereas Desperate Housewives was the least popular

- If you want to be really fancy, you might be able to say things like: "roughly 3 times as many teachers preferred Lost to Desperate Housewives"

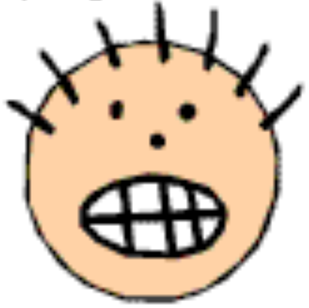
4. What CAN'T we tell from Pie Charts

- Well, imagine we were just given our pie chart (and no original data), and someone said: "how many maths teachers said that Countdown was their favourite show?", what would we say?...

- Well, probably not a lot, because there is no way of knowing!

- Unless we are told how many people were surveyed all together, we cannot answer that question!

- When making statements based on Pie Charts, just make sure what you are saying is definitely, 100% true!



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5. Interpreting Pie Charts

Big Example 2

240 Maths teachers were asked "what is your favourite drink?" and a pie chart was drawn to show to information.

Work out how many teachers preferred coffee

To answer this question we must do the opposite of what we did when we were drawing the pie chart - we must use our angles to find our totals!

Let's look at coffee... it takes up 84^0 out of 360^0, and what we want to know is "how much does it take up out of our 240 people?"

Well, what about using this as an excuse to show off our Algebra skills!...

$$\frac{84}{360} = \frac{?}{240} \xrightarrow{\text{Multiply both sides by 240}} \frac{84}{360} \times 240 = ?$$

So, turning to our calculator, we get an answer of... 56 people

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6. Stem and Leaf Diagrams

Why do we bother with Statistical Diagrams?

- The answer to this question is similar to the one for: "why do we bother working out averages and measures of spread?".

- We live in a world jam-packed full of statistics, and if we were forced to look at all the facts and figures in their raw, untreated form, not only would we probably not be able to make any sense out of them, but there is also a very good chance our heads would explode.

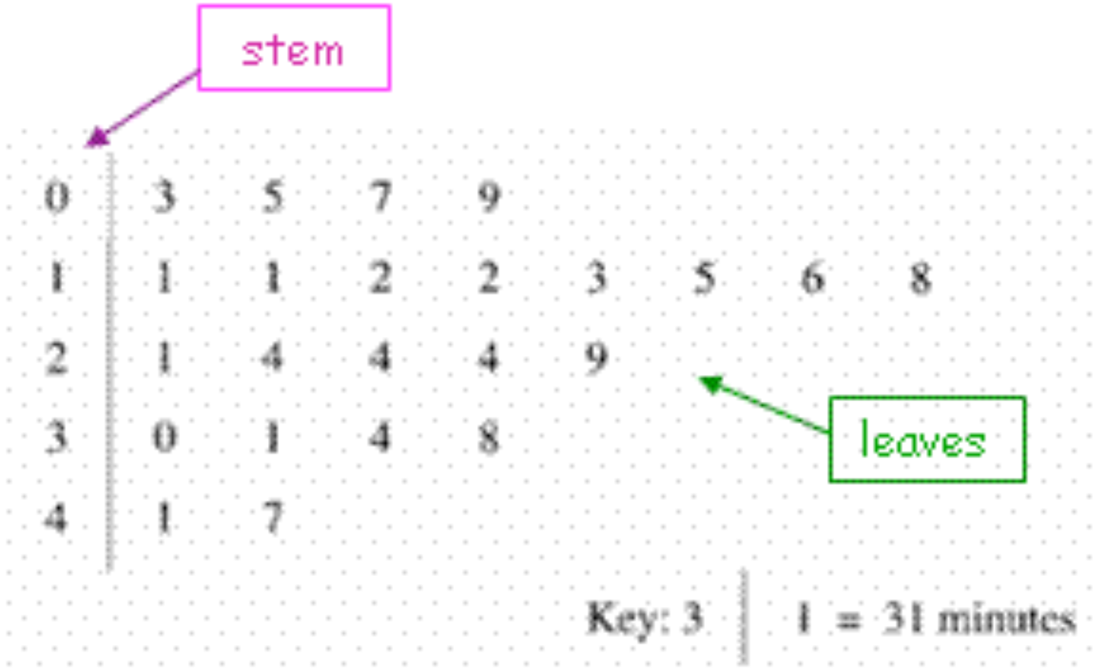
- Statistical Diagrams - if they are done properly - present those figures in a clear, concise, visually pleasing way, allowing us to make some sense out of the figures, summarise them, and compare them to other sets of data.

1. What are Stem and Leaf Diagrams

- To be honest, Stem and Leaf Diagrams are just a fancy way of listing a fairly large group of numbers in order

- They are seen as a quicker, more convenient, and ultimately more useful way of presenting data than just a long list of numbers.

- An example of a typical Stem and Leaf Diagram is on the right



0	3	5	7	9				
1	1	1	2	2	3	5	6	8
2	1	4	4	4	9			
3	0	1	4	8				
4	1	7						

Key: 3 | 1 = 31 minutes

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Big Example

Here are the times (in minutes) that it takes Mr Barton to actually get out of bed after his alarm has sounded on a Monday morning:

12 6 20 24 52 41 3 35 55 32 11 13 2 25 38 39 41 52 13 59 18 22 29 35

Use the data to construct a Stem and Leaf Diagram, and then calculate the Median and Inter-Quartile Range

2. Constructing a Stem and Leaf Diagram

1. Decide on your stems - these are the digits which go down the left hand side of your diagram. You should choose them in a way so that you have between 4 and 10 groups, and so that each of your leaves is only one digit!

2. Go through your data, in the order in which it is written, and add it to the correct stem on your diagram. I would mark each piece of data once it has been entered, so you don't loose your spot!

3. When completed, this is your un-ordered stem and leaf diagram

4. Now draw yourself another stem and leaf diagram, but this time put the leaves in order!

Note: Everyone seems to want to jump straight to the ordered stem and leaf diagram, but I promise you this way is quicker and a lot safer in terms of mistakes!

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Our Example

1. For our stems, we only need the first digit of each piece of data, and I think 6 groups should do us!

Note: to make sure we can enter the single digit pieces of data, we must make our first stem start with 0

0	
1	
2	
3	
4	
5	

2. Next we begin to go through our data, creating them leaves of our diagram, marking off each piece of data as we use it...

12 6 20 24 52 41 3 35 55 32 11 13 2 25 38 39 41 52 13 59 18 22 29 35

0	6
1	2
2	0
3	
4	
5	

3. Continuing like this eventually gives us our un-ordered stem and leaf diagram

0	6	3	2		
1	2	1	3	3	8
2	0	4	5	2	9
3	5	2	8	9	5
4	1	1			
5	2	5	2	9	

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Things to Notice:

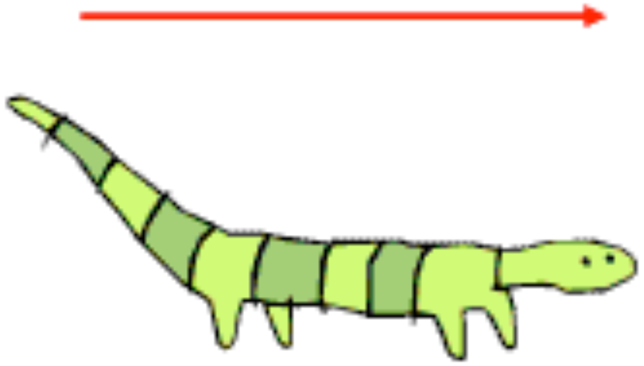
a) For numbers like 20, we must place a 0 as our leaf

b) Single digit numbers are placed on the top stem

c) If we come across a number that we have already recorded, we must record it again!

4. Having made sure that we have 24 digits as the leaves of our diagram (there are 24 pieces of data!), we can now very quickly change our unordered diagram into an ordered one by placing the leaves on each stem… in order!

0	6	3	2		
1	2	1	3	3	8
2	0	4	5	2	9
3	5	2	8	9	5
4	1	1			
5	2	5	2	9	



0	2	3	6		
1	1	2	3	3	8
2	0	2	4	5	9
3	2	5	5	8	9
4	1	1			
5	2	2	5	9	

Key

2 | 0 = 20

5. And the final thing we must remember to add to our diagram is a KEY!

You must let anyone who looks at your diagram know exactly what each of the leaves stands for… so in this case, I have chosen the 2 and the 0 from the 3rd row, and just explained that this is actually 20!

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3. Finding the Median and Inter-Quartile Range from a Stem and Leaf Diagram

Remember: our Stem and Leaf Diagram is just a group of numbers, written out in order… and so we don't have to learn any different skills to find the median and inter-quartile range!

(a) Finding the Median

It's the usual thing… the median is the middle number… and if there is an even amount of numbers, then you will have two numbers in the middle.

Draw a box around the number/numbers you think are in the middle, and make sure you have the same amount of numbers on either side!

0	2	3	6		
1	1	2	3	3	8
2	0	2	4	5	9
3	2	5	5	8	9
4	1	1			
5	2	2	5	9	

Check: There are 11 numbers to the left of our box, and 11 numbers to the right!

Our median is half way between our two numbers in the box, and so is… 27!

So many people would put 7… but remember what the leaves actually stand for!

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(b) Finding the Inter-Quartile Range

Remember: Inter-Quartile Range = Upper Quartile - Lower Quartile

The way I do this is to think of the Lower Quartile as being the median of the lower half of numbers... and the Upper Quartile as the median of the upper half of numbers.

And I just find these values the same old way... using my boxes, and making sure there are the same amount of numbers on either side!

0	2	3	6		
1	1	2	$\boxed{3 \quad 3}$		8
2	0	2	4	5	9
3	2	5	5	8	$\boxed{9}$
4	$\boxed{1}$	1			
5	2	2	5	9	

$$\frac{13+13}{2} = 13$$

Lower Quartile = 13

(5 numbers either side in the lower half)

$$\frac{39+41}{2} = 40$$

Upper Quartile = 40

(5 numbers either side in the upper half)

Inter-Quartile Range = 40 - 13
= 27

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4. What's GOOD about Stem and Leaf Diagrams?

- Well, the major advantage over things like bar charts and histograms, is that no information is lost – the stem and leaf diagram keeps and allows you to see each original piece of data

- It is also quite an effective way of ordering and displaying relatively small sets of data

5. What's BAD about Stem and Leaf Diagrams?

- Well, it's quite time consuming, and impractical for large data sets. Imagine how long it would take to sort over 300 pieces of data, and how complicated the final diagram would look!

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7. Bar Charts and Histograms

Why do we bother with Statistical Diagrams?

- The answer to this question is similar to the one for: "why do we bother working out averages and measures of spread?".

- We live in a world jam-packed full of statistics, and if we were forced to look at all the facts and figures in their raw, untreated form, not only would we probably not be able to make any sense out of them, but there is also a very good chance our heads would explode.

- Statistical Diagrams – if they are done properly - present those figures in a clear, concise, visually pleasing way, allowing us to make some sense out of the figures, summarise them, and compare them to other sets of data.

Big Example

To the right is a table showing the length of applause after Mr Barton announces that there will be no homework tonight. Construct a Bar Chart and a Histogram, and comment on the differences

Length of Applause (mins)	Frequency
$0 < a \leq 1$	2
$1 < a \leq 2$	4
$2 < a \leq 3$	15
$3 < a \leq 5$	10
$5 < h \leq 8$	6
$8 < h \leq 13$	5

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1. Drawing a Bar Chart (Frequency Diagram)

Note: Sometimes bar charts are called Frequency Diagrams!

1. Decide on an appropriate scale to fit the paper you are working with - as a general rule, the bar chart (or any statistical diagram, for that matter) should take up between half and three-quarters of the space you have to work with.

Crucial: Your numbers must go up in equal steps!... see 8. Scatter Diagrams for examples of some very dodgy scales!

2. Label your axes. Is it usual to put frequency (total) on the y axis, and whatever the data is along the x axis.

3. Carefully draw in your bars... and add a title!

Note: In examples like this where the groups are numbers, then it is usual to have the bars touching each other

If the groups were categories (such as "colour of cars"), then you could have gaps between your bars if you like!

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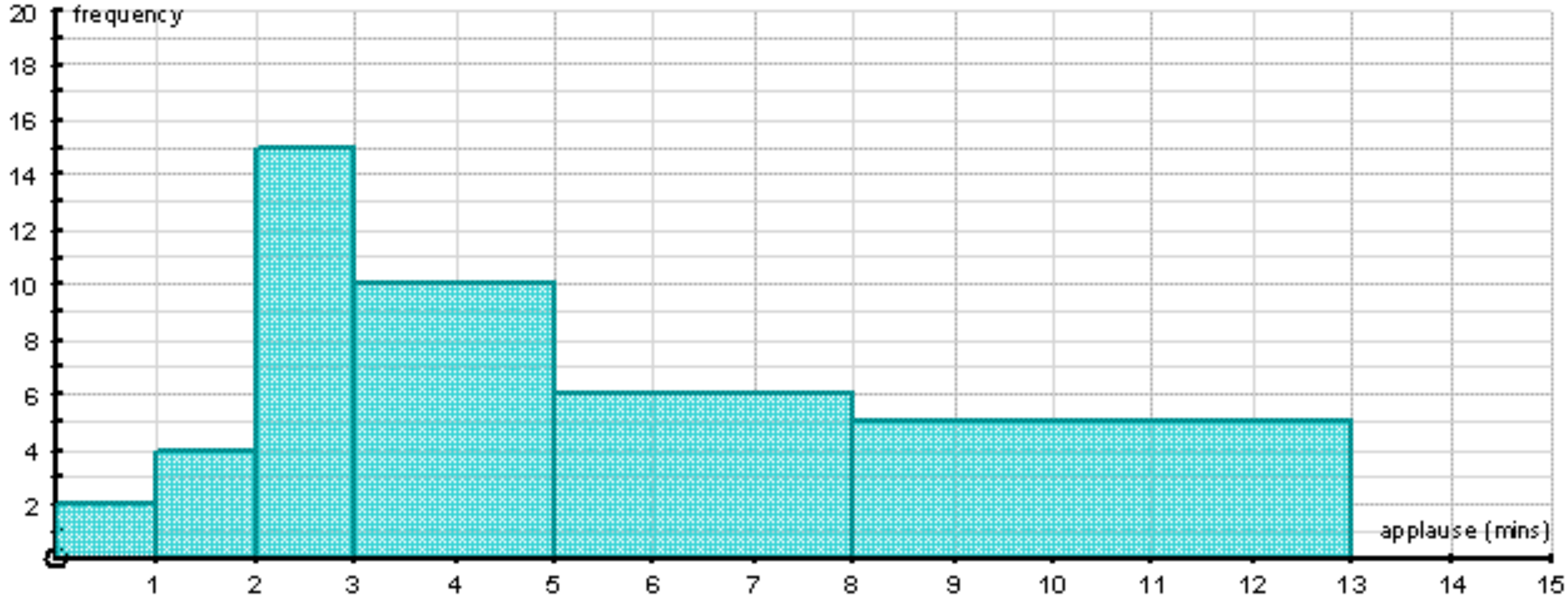
A Bar Chart to show the Length of Applause after Mr Barton says "no homework"



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2. Drawing a Histogram

The major difference between a bar chart and a histogram is what goes on the y axis

On a Bar Chart it is Frequency

On a Histogram it is... Frequency Density!

The reasons for this will be discussed soon!

1. Add two extra columns to your table... Group Width and Frequency Density

2. To work out Group Width, we just do the upper limit of each group minus the lower limit

3. To work out Frequency Density, we use this lovely formula:

$$\text{Frequency Density} = \frac{\text{Frequency}}{\text{Group Width}}$$

4. We then plot frequency density on the y axis, and our data on the x axis as before

Note: In Histograms you have no choice... the bars must always be touching!

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Group Width: 1 – 0 = 1

Frequency Density:

2 ÷ 1 = 2

Length of Applause (mins)	Frequency	Group Width	Frequency Density
$0 < a \le 1$	2	1	2
$1 < a \le 2$	4	1	4
$2 < a \le 3$	15	1	15
$3 < a \le 5$	10	2	5
$5 < h \le 8$	6	3	2
$8 < h \le 13$	5	5	1

Group Width: 13 – 8 = 5

Frequency Density:

5 ÷ 5 = 1

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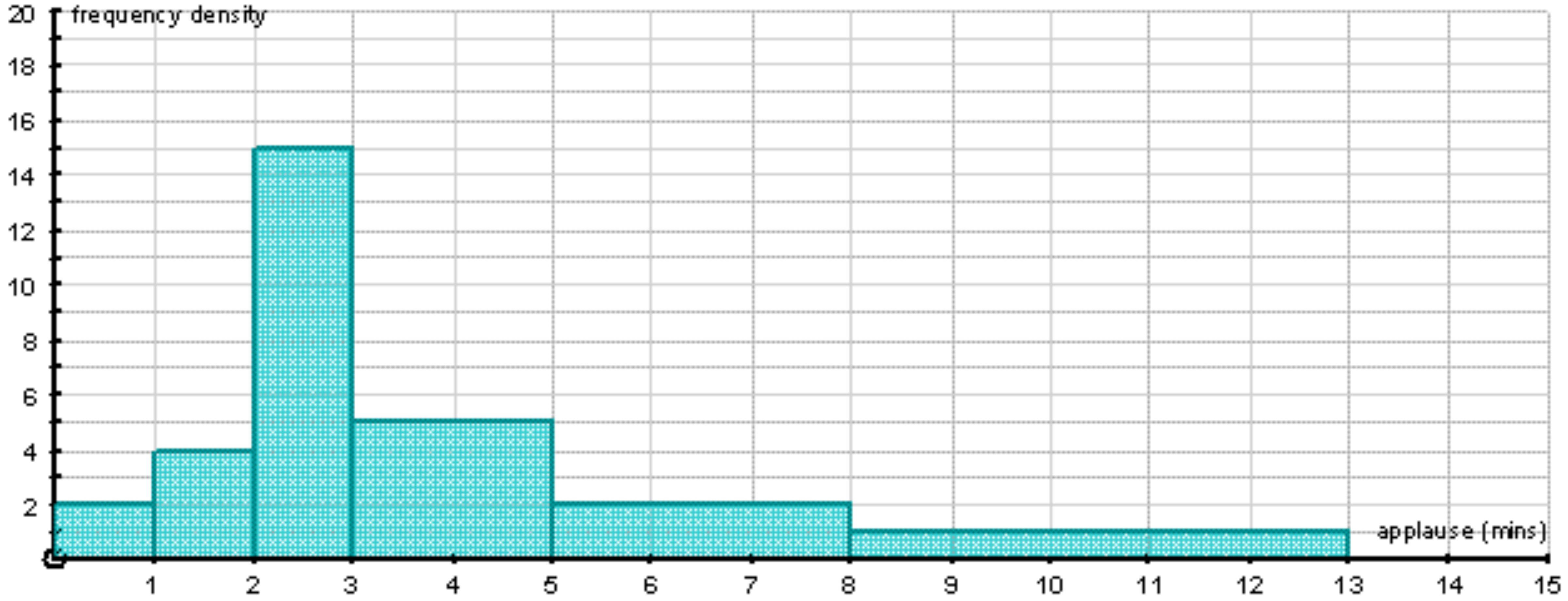
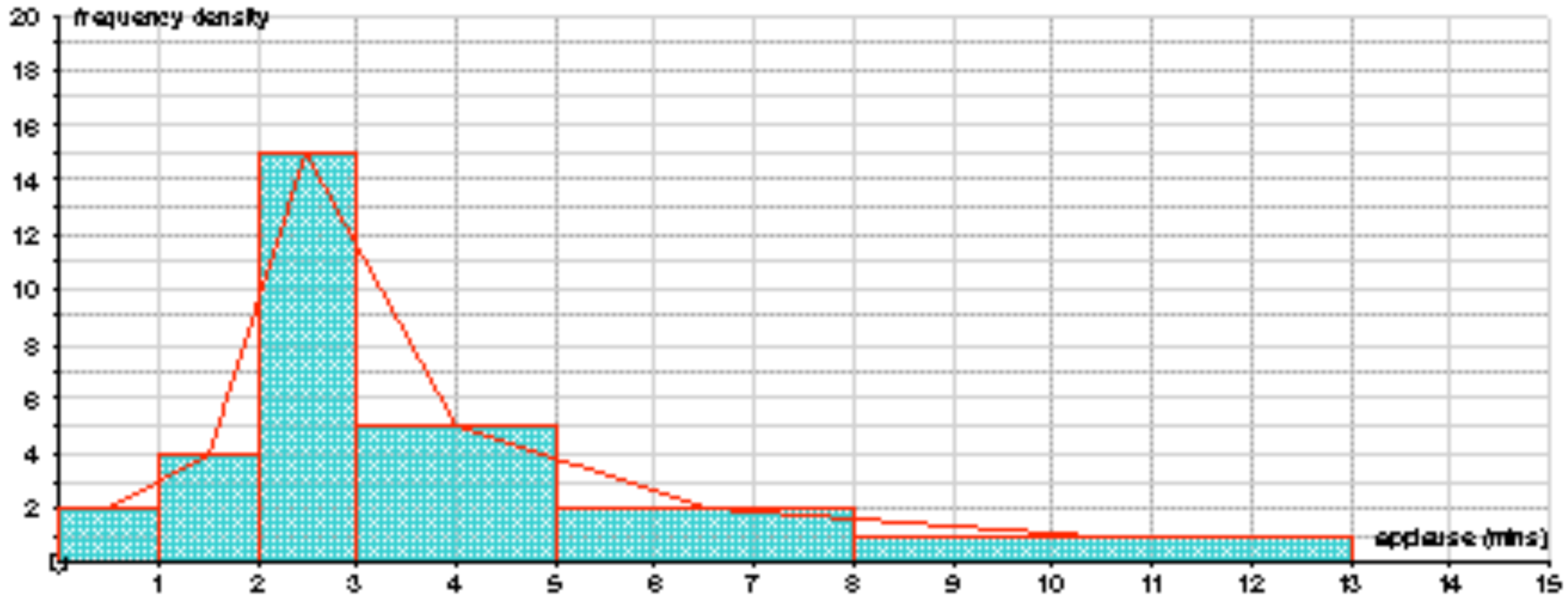
A Histogram to show the Length of Applause after Mr Barton says "no homework"



Note: Sometimes you get asked to draw a Frequency Polygon. Do not fear, this is just a Histogram, but with the top mid-points of each bar joined together with straight lines!



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3. What is the Point of Histograms?

- Have a quick look back at the two diagrams
- On the Bar Chart, which group looks like it has the most people in it?... maybe the last one?... because it takes up such a large area compared to the other groups... but if you look on the table of data, this group only had a total of 5!

  So... Bar Charts can be deceptive
- But look at the Histogram... the group that looks the biggest now is the "between 2 and 3 minutes" group... and this is the group that has the highest frequency!
- The reason for this is that in a Histogram, the areas of the bars are proportional to the frequency, and not just the height like a bar chart.
- Note: If all our groups had the same width, it wouldn't matter, but often they do not, so that is why Histograms tend to be used more than Bar Charts!

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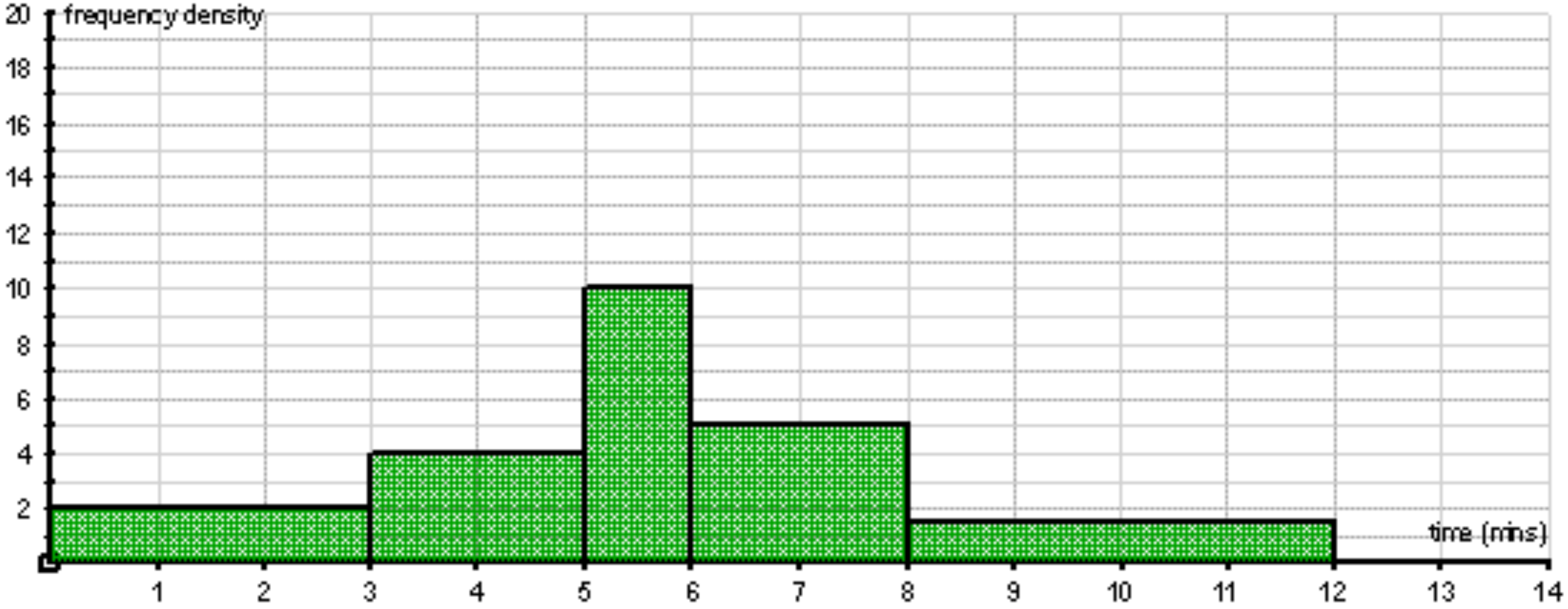
4. Interpreting Histograms

Example: Here is a Histogram showing the time taken by some year 7s to complete all their times tables. Find the frequencies of each group.



This time we are given Frequency Density, and asked to work out Frequency… well, if we do a little re-arranging to our formula we get…

Frequency = Frequency Density x Group Width

Time (mins)	Frequency Density	Group Width	Working	Frequency
0 < t ≤ 3	2	3	2 × 3 = 6	**6**
3 < t ≤ 5	4	2	4 × 2 = 8	**8**
5 < t ≤ 6	10	1	10 × 1 = 10	**10**
6 < t ≤ 8	5	2	5 × 2 = 10	**10**
8 < t ≤ 12	1.5	4	1.5 × 4 = 6	**6**

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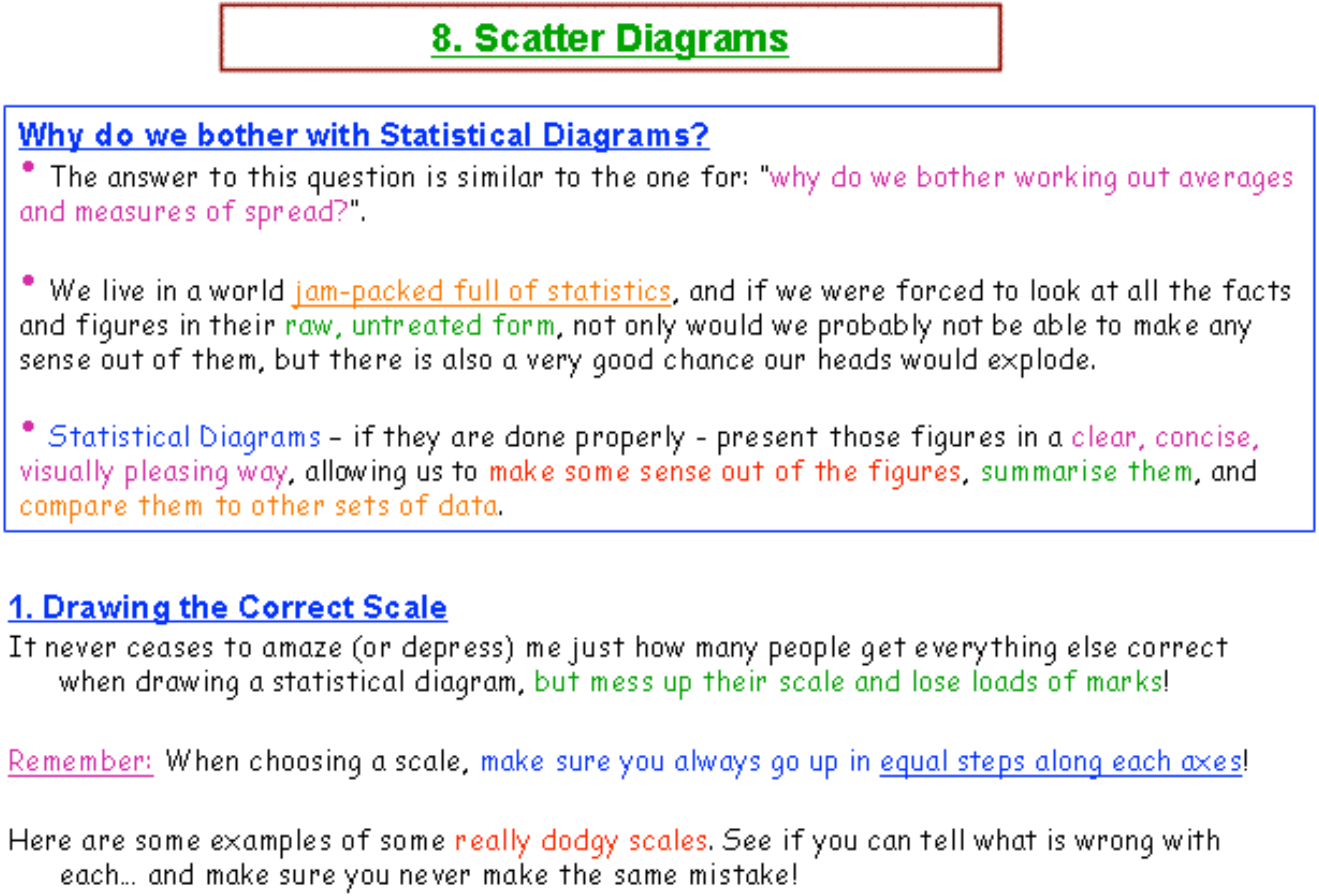

8. Scatter Diagrams

Why do we bother with Statistical Diagrams?

- The answer to this question is similar to the one for: "why do we bother working out averages and measures of spread?".

- We live in a world jam-packed full of statistics, and if we were forced to look at all the facts and figures in their raw, untreated form, not only would we probably not be able to make any sense out of them, but there is also a very good chance our heads would explode.

- Statistical Diagrams – if they are done properly - present those figures in a clear, concise, visually pleasing way, allowing us to make some sense out of the figures, summarise them, and compare them to other sets of data.

1. Drawing the Correct Scale

It never ceases to amaze (or depress) me just how many people get everything else correct when drawing a statistical diagram, but mess up their scale and lose loads of marks!

Remember: When choosing a scale, make sure you always go up in equal steps along each axes!

Here are some examples of some really dodgy scales. See if you can tell what is wrong with each… and make sure you never make the same mistake!

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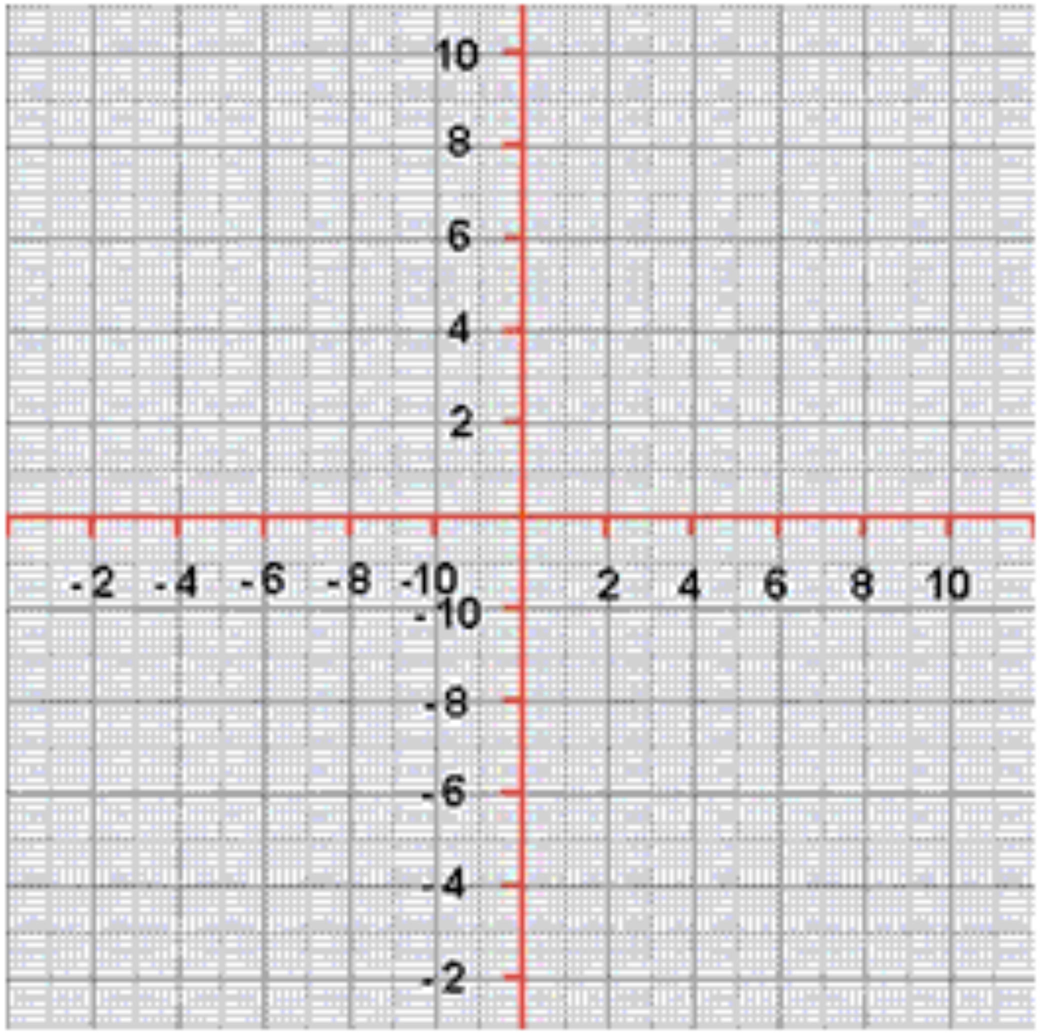
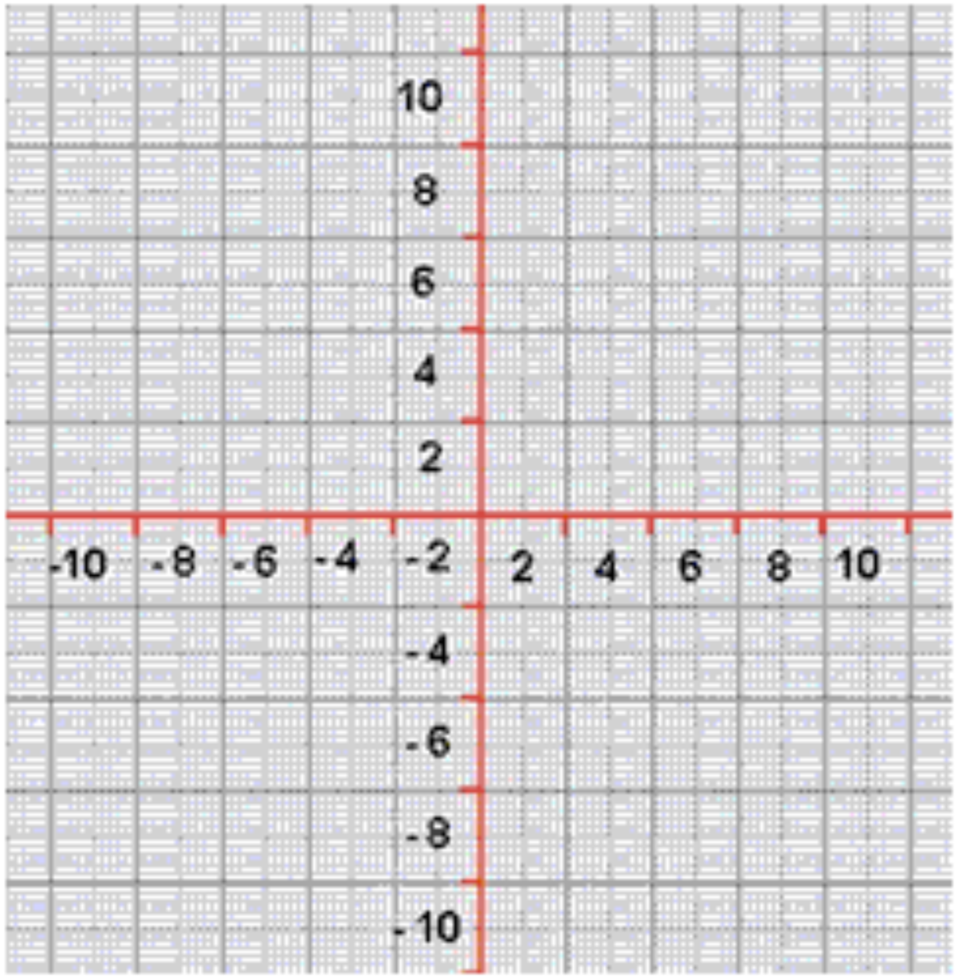
Example 1



Example 2



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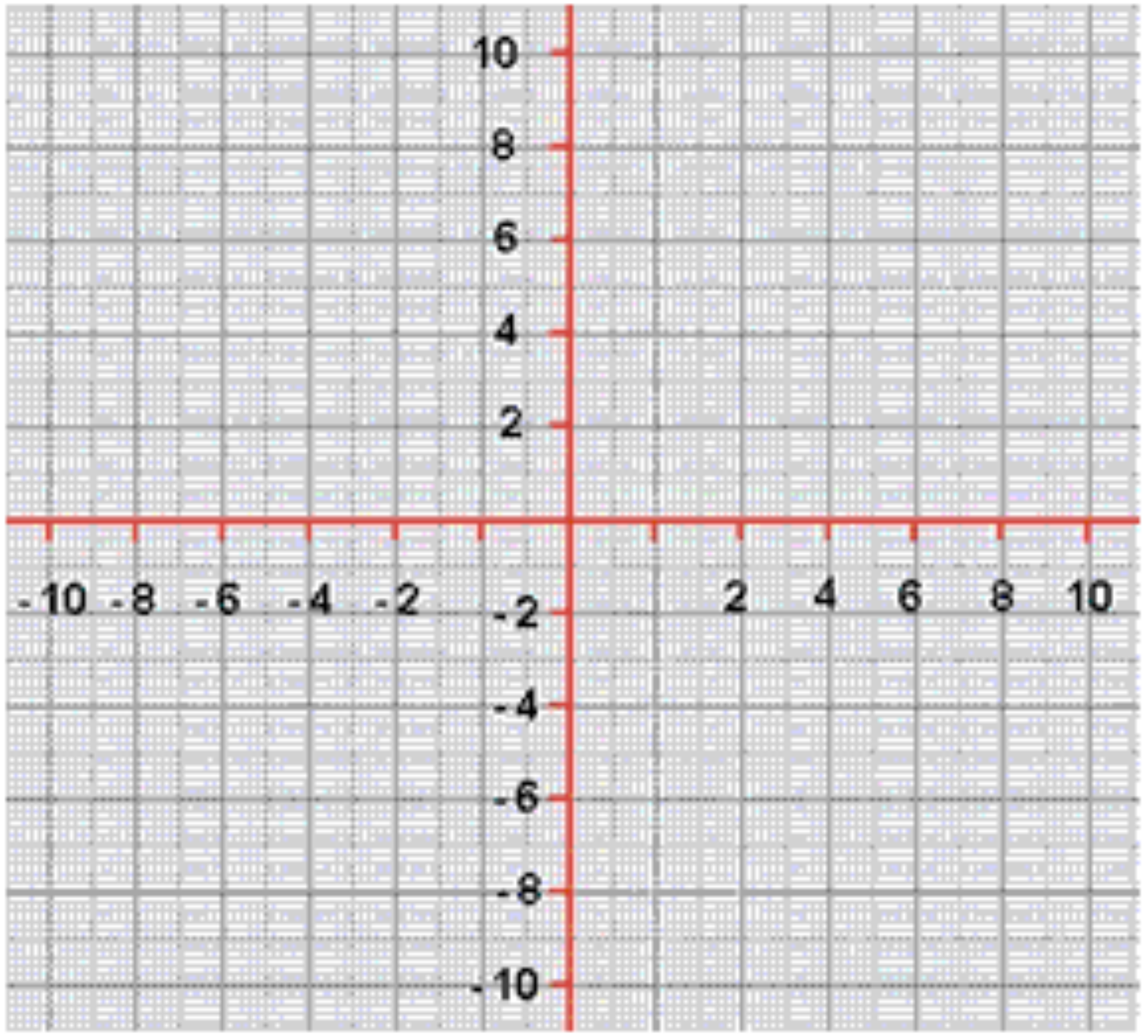
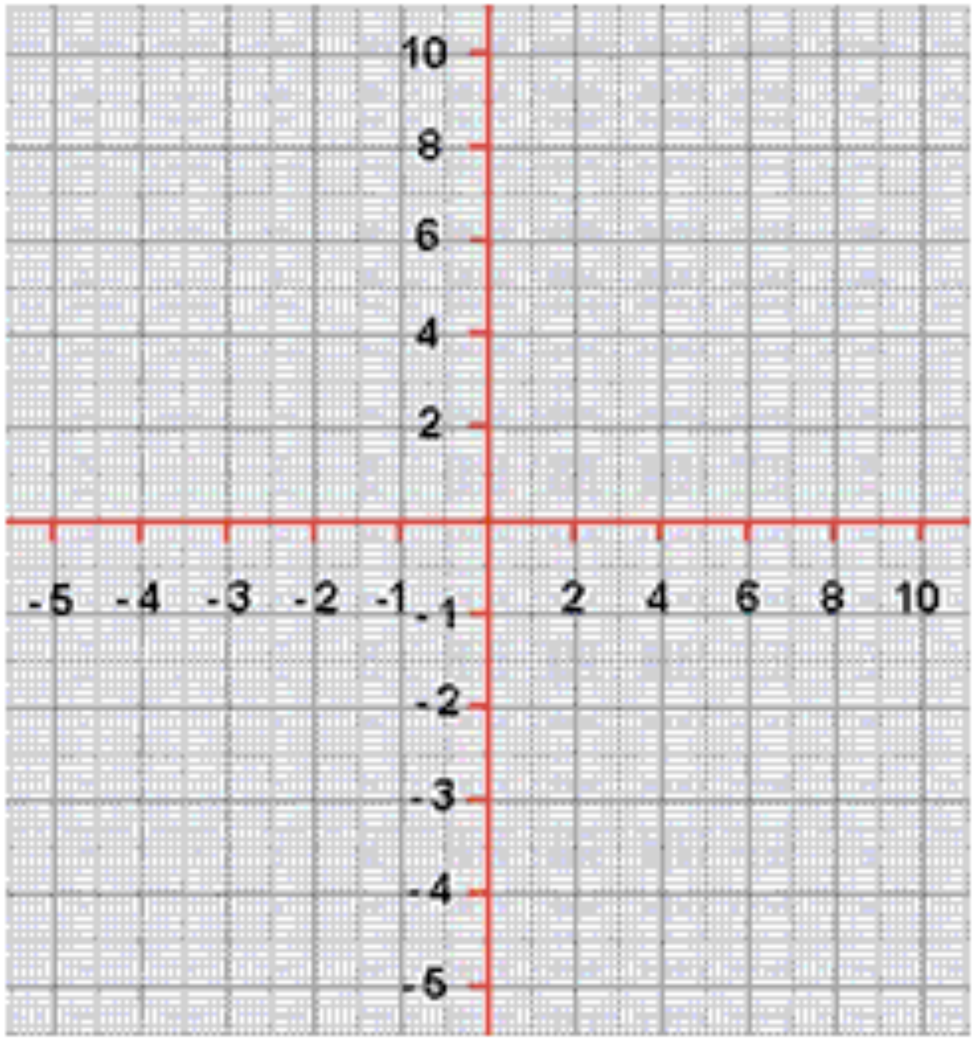
Example 3



Example 4



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The Answers

1) This person has messed up their negative numbers. Remember, scales must go from smallest to biggest, from left to right, and down to up.

2) Classic mistake. Numbers must go on the lines, not between the spaces!

3) How many times have I seen this? The spaces around the centre (origin) are not equal. Look at the gap between 2 and - 2…. Deary me!

4) Inconsistent scales! Notice the numbers go up by 1 in the negatives and then 2 in the positives!

Note: Another mistake in all of the diagrams is that the x and y axes are not labelled!

Big Example

Below is a table showing the number of pupils who fail to hand in their maths homework each day, and the minutes of yoga I need to do to calm myself down

Pupils missing homework	3	5	2	10	2	0	4	8	15	6	1	4
Minutes of yoga	10	12	9	25	8	3	15	20	26	10	7	10

Draw a scatter diagram to show the information, add a line of best fit, and comment on the correlation

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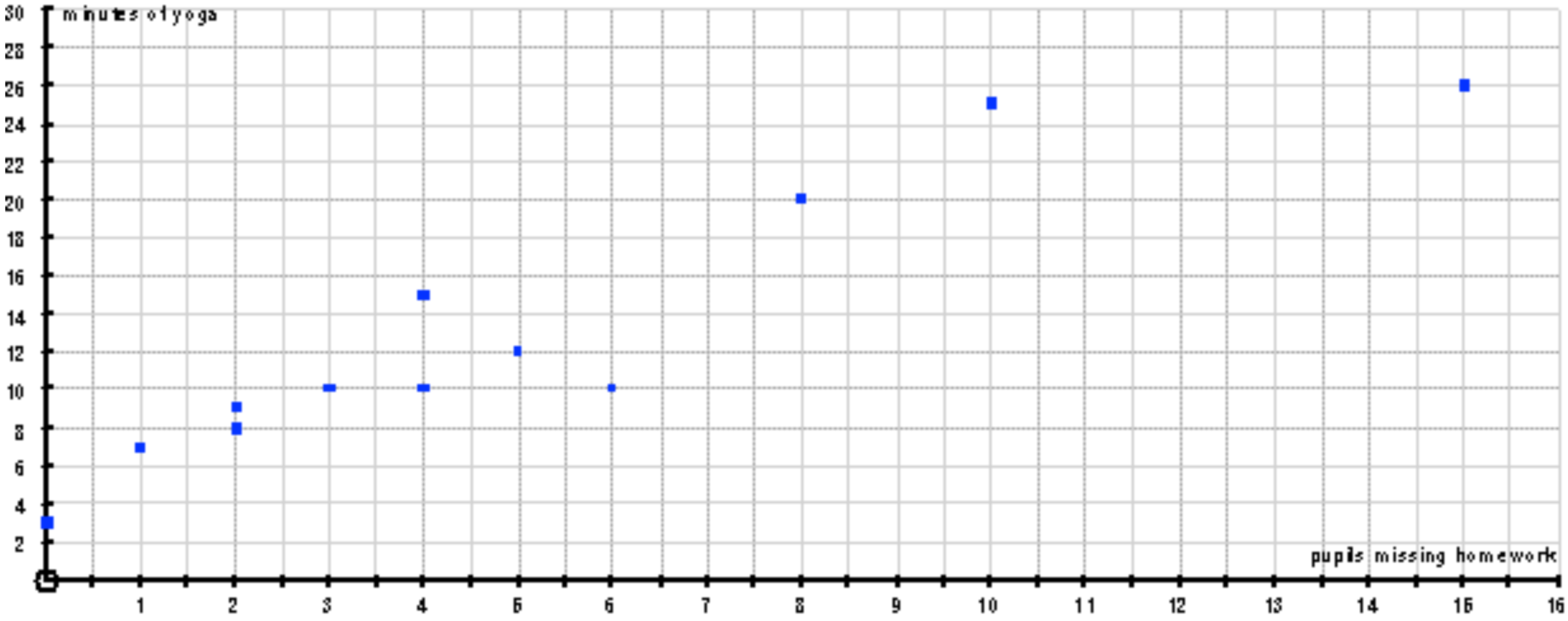
2. Drawing a Scatter Diagram

1. Decide on an appropriate scale that will look a decent size and fit all the data in!

Note: It doesn't really matter which set of data goes on the x axis and which on the y... but personally I like to put the one with the biggest numbers on the y axis! BUT: remember to label both axes, including units!

2. Carefully mark each piece of data on your diagram with a dot/cross, and when you have finished, check you have the correct number of crosses!



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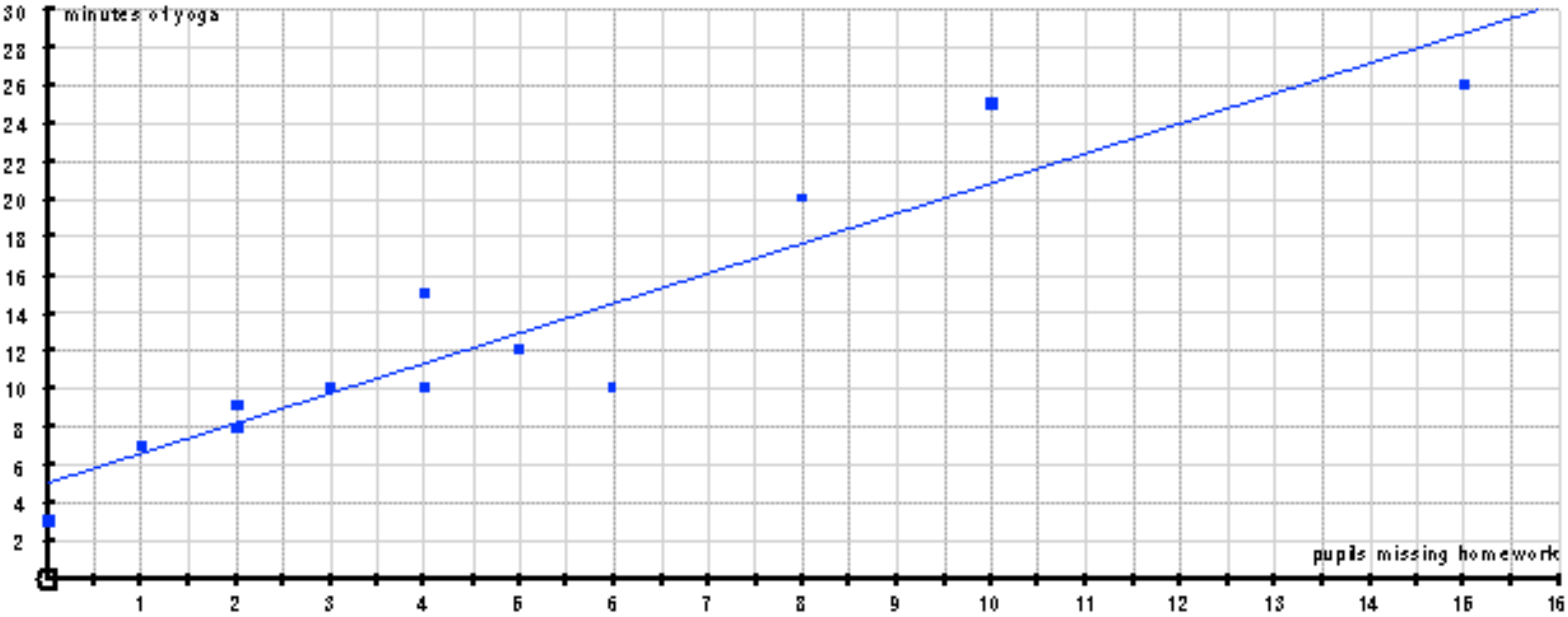
3. The Line of Best Fit

This is a single straight line which is supposed to be a good representation of the pattern / trend of the data

Tips for drawing it:

- Try to get roughly the same amount of points above the line as below
- Experiment by using your ruler as your line, and only draw the line in when you are happy
- Don't spend too long deciding, and don't try to make it perfect!

Note: Your line does NOT have to start at the origin (0, 0)



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4. Correlation

The most important use of scatter diagrams is to determine the type (if any) of correlation between two variables

Correlation is just a posh word for relationship.

There are two categories of correlation that you need to be familiar with:

DIRECTION

Positive – line slopes upwards

As one variable increases, so does the other

Negative – line slopes downwards

As one variable increases, the other decreases

No correlation – line is close to horizontal

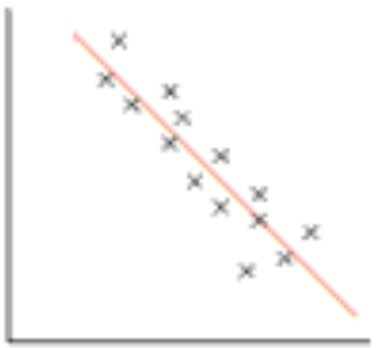
No relationship between the variables

STRENGTH

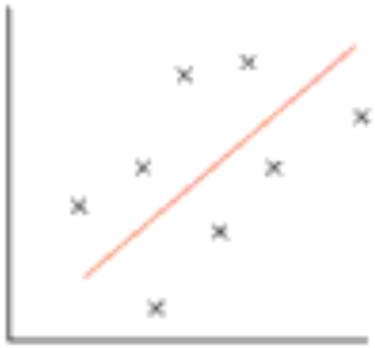
Strong – dots are close to each other

Weak – dots are far apart

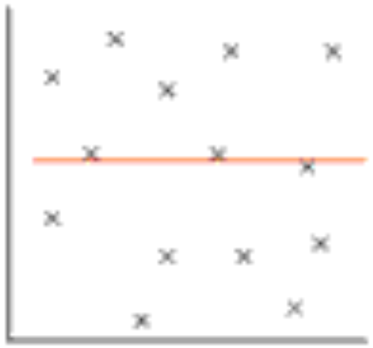
Tip: When deciding on the strength of correlation, I have a little rule: "the longer it takes me to decide where to draw the line of best fit, the weaker the correlation"



Strong negative



Weak Positive



No Correlation

Looking at our example, I would say there is a fairly strong, positive correlation. This is no surprise, because as the number of missing homeworks increases, so to does my need for yoga!

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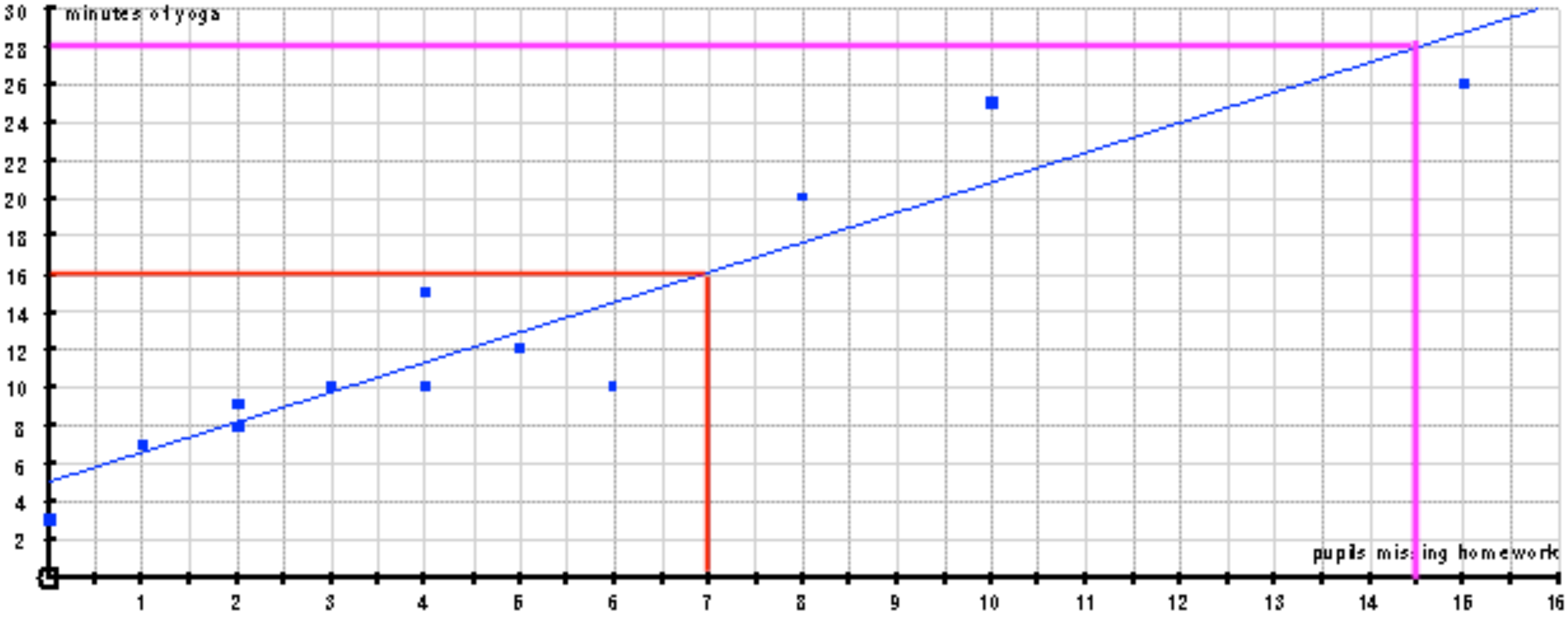
5. Using your Line of Best Fit

Once we have drawn our line of best fit, we can use it to predict results we don't already have

Note: The stronger the correlation, the more reliable these predictions will be!



Question 1: If 7 pupils forget to hand in their homework, how many minutes of yoga might Mr Barton do?

Question 2: If Mr Barton does 28 minutes of yoga, how many pupils might have forgotten their homework?

Following the red line up and across gives...

16 minutes

Following the purple line across and down...

14.5 pupils

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